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An investigation of the formation of plastic hinges in simple structures

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AN INVESTIGATION OF THE FORMATION OF PLASTIC HINGES IN SIMPLE STRUCTURES

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JAMES I. GIBSON

AN INVESTIGATION

OF THE

FORMATION OF PLASTIC HINGES

IN

SIMPLE STRUCTURES

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by

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and

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INTRODUCTION

OBJECT

The object of this thesis is twofold: first, we seek to develop and propose a simplified method of designing redundant structures wherein the only design criterion is that the structure will safely resist failure. Second, we seek to show by the results of experimental tests that the proposed method gives an accurate picture of the true strength of welded redundant structures. It is not our purpose to condemn the elastic working stress theory as "too conservative" or to propose the method developed herein as a "cure all". Nor do we wish to imply that the experimental results are of sufficient scope to allow the method to be applied to all design problems. Indeed, since method and application are generally regarded as separate and distinct fields, and since our primary purpose is to illustrate a method, we feel that we will do well to say as little as possible about applications. We do, however, believe that the method of assumed plastic hinges offers a simple means of selecting a true factor of safety and provides a better measure for the margin of safety in a redundant structure than does the elastic working stress theory. The experimental tests show that the method yields very accurate results for the structures tested, although we admit that perhaps subsequent tests will reveal unforeseen weaknesses. We are of the opinion, however, that any

method which gives a better measure of the real useful strength of a structure for general cases will always give results closer to what they should be in individual cases. Such a method is the subject of this work.

SCOPE

As stated in the title, the subject of this thesis is an investigation of the formation of plastic hinges in simple structures. The scope of this investigation was of necessity limited by both time and the lack of a suitable static loading machine. Given an unlimited amount of time, we might have undertaken the design and construction of a static loader capable of subjecting a test model to a great variety of loading conditions. But since, as we pointed out in stating our object, our primary purpose is to illustrate a method and not to determine applications, we felt that our purpose might better be served by selecting simple problems which would indicate the validity of the proposed design method more decisively. We therefore decided to limit our investigations to continuous beams and to simple rigid frames. We also limited our investigations to concentrated static loads, with only one load applied to each span or frame. Furthermore, since the ability to predict the absolute value of the plastic (or limiting) moment was regarded as incidental to the ability to predict the loads required to form plastic hinges, it was decided to use models of

rectangular cross-section in the interests of economy and simplicity. The details of designing and building the testing apparatus and of the method of making the models are covered in the main body of this work. It seems sufficient to state here that both the apparatus and the models proved entirely adequate to substantiate the theory within the bounds of the assigned scope.

THESIS

For the last one hundred and fifty years, elastic stress analysis has been so intimately allied to the theory of strength that the two have come to be regarded as almost synonymous. During this period of time, and especially in recent years, we have seen the field of construction engineering emerge from an era of statically determinant construction into an era of statically indeterminant or redundant construction. Certain conventions, rules, and practices, developed during the former period largely as a result of elastic stress analysis reasoning, have been carried over into the latter period. Many of these rules and values are definitely open to question. They involve assumptions which are little more than wishful thinking. When applied to statically indeterminant structures, they appear to be quite inadequate.

WEAKNESS OF ELASTIC STRESS THEORY

It seems that the most satisfactory method of demonstrating the weakness of the elastic stress theory when applied to statically indeterminant structures is to cite a simple example: Let us suppose that we are given a continuous beam of two equal spans loaded with a uniformly distributed load. The bending moment over the central support is $wL^2/8$. Now let us replace the continuous beam by two simple beams placed over the

same supports and again uniformly loaded. The bending moment midway between the supports is $wL^2/8$ and is thus equal to the bending moment over the central support in the case of the continuous beam. If the elastic working stress is taken as a criterion of strength, then the implication is unmistakeably to the effect that the strength of the continuous beam is identical with that of the two simple beams. In effect, the elastic-working-stress concept argues that the continuous beam may be replaced by two simple beams without its load carrying capacity being effected. Experienced engineers will refuse to accept the contention that the continuous beam may be cut over the central support without impairing its load carrying capacity. Horse sense is, of course, not infallible but in this case it seems to be superior to the elastic strength theory.

BASIC CONSIDERATIONS

Any theory is based on assumptions or premises. The theory of strength is predicated on two basic considerations: first, that of equilibrium; and second, that of continuity. The first is by far the more important of the two. It is expressed conventionally by $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. Any violation of the laws of equilibrium, no matter how minor, results in collapse. The considerations of continuity are secondary to those of static equilibrium and serve to supplement

the latter in arriving at a picture of the strength of a given structure. To satisfy the conditions of continuity, we have a choice of two theories. One presupposes elastic behavior and elastic stress distribution while the other presupposes ductile or semiductile stress distribution. The former is known as the theory of elasticity while the latter is known, though not so generally, as the theory of limit design. The theory of elasticity argues that the primary criterion of strength is the elastic working stress. The theory of limit design argues that the primary criterion of strength is deformation. The theory of elasticity is well known to all structural engineers and needs no review here. The theory of limit design is less familiar and so a description of its application will be presented after a brief review of the characteristics of mild steel.

DUCTILITY

Ductility is the property of a material by virtue of which it deforms extensively under a constant or slightly increasing stress. Ductility is most ideally exemplified by mild steel such as is used in most hot rolled structural shapes. Figure 1 represents the ideal stress-strain curve for a typical mild steel. The portion AB of Figure 1 represents purely elastic behavior of mild steel. When a stress corresponding to A' is reached (from 36,000 to 40,000 psi in the case of

mild steel), the material suddenly yields an amount represented by BC, figure 1. The distance BC, which represents yielding under a constant stress, is 10 to 20 times as large as the purely elastic deformation represented by A'B. After a strain corresponding to point C is reached, progressive straining takes place under slowly increasing stresses. The maximum strain is represented by point D. The significant fact to realize is that the distance from A'' to D is 200 to 300 times as large as the distance from A' to B while the distance from A to A'' is generally taken as two times the distance from A to A'. This property of ductility, excessive deformation under substantially constant stress, is generally recognized as a highly desirable property. In view of the great importance that this property of ductility unquestionably holds, it is strange that up to the present time it has received but scant attention in any formal theory directed to the study of the strength of structures. Of recent years, the word plasticity has been commonly used in the sense of ductility. In this thesis ductility will refer to the property of the material in question by virtue of which ductile flow is able to take place. Plasticity will refer to the ductile behavior of a structural member in a region where complete ductile flow has occurred. Thus, in a sense, ductility is a

ensue and plasticity is in effect.

LIMIT DESIGN

The application of the theory of limit design can best be shown by considering the simple beam in figure 2(a) with a concentrated load at the midpoint. The maximum value of the load P for which the elastic-limit stress, s_1, is just reached in the outer fibers under the load is found from the equation $s_1 = Mc/I = 3P_1L/2bh^2$ from which $P_1 = 2s_1bh^2/3L$. The stress distribution over the cross section of the beam will then be as shown in figure 2(b). If the elastic-ductile properties of the material correspond to curve ABC, figure 1; further, if a plane before bending remains a plane after bending in any elastic part of the beam, then the stress distribution under a load P_2 slightly in excess of P_1 will be as shown in figure 2(d). The limiting resisting moment of the beam will be developed when all the fibers over the entire cross section are stressed with their elastic limit stress s_1. The stress distribution under the load P_3 will be as shown in figure 2(f). Let figure 2(g) represent the cross section on which this limit stress distribution has been reached. From $F_x = 0$, we derive $A_1s_1 = A_2s_1$ and hence $A_1 = A_2$. Thus it will be seen that in an unsymmetrical beam, the neutral axis does not necessarily pass through the centroid of the cross section. The bending moment resulting from the

FIGURE 1

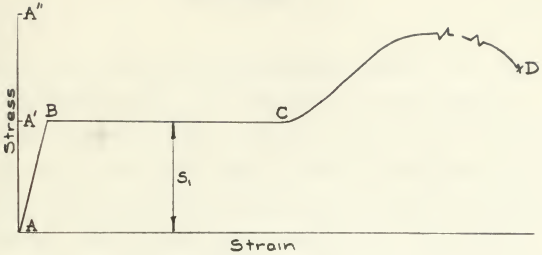
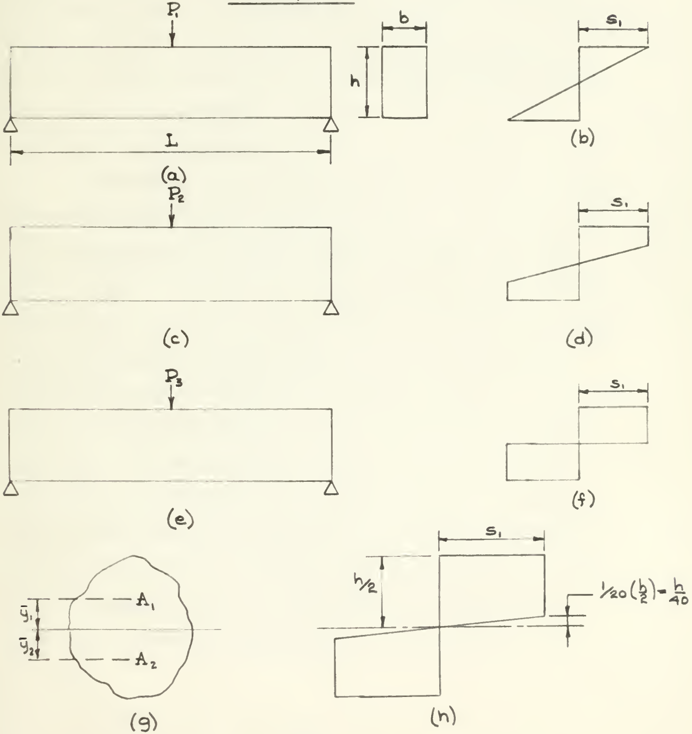


FIGURE 2



stress distribution of figure 2(f) would be:
$M = s_1(A_1\bar{y}_1 + A_2\bar{y}_2)$. In the case of a symmetrical beam, the neutral axis passes through the centroid of the section and $A_1\bar{y}_1 = A_2\bar{y}_2$. In this case $M = 2s_1A\bar{y}$ where $A\bar{y}$ represents the static moment about the neutral axis of a part of the cross section area which lies to one side of the neutral axis. Thus for the rectangular in figure 2(a), $M = 2s_1(bh/2)(h/4) = s_1bh^2/4$ and $P_3 = s_1bh^2/L$. Thus the limit load carrying capacity of a rectangular simple beam is 50% greater than its elastic load carrying capacity.

EFFECT OF ELASTIC CORE

It may seem unreasonable to assume that the elastic core near the neutral axis ever vanishes completely. If the distance A'C of figure 1 equals 20(A'B), then the limit stress pattern would appear as shown in figure 2(h) rather than as shown in figure 2(f). The difference between the two figures would have an extremely small effect on the corresponding resisting moment. In the more common WF and I beams, the limit resisting moment is only 6 to 10% in excess of the elastic resisting moment and any effects of an elastic core are of even lesser magnitude than in the case of rectangular beams. Since most designs do employ the more economical WF section, it may appear that the use of the formula $M = 2s_1A\bar{y}$ is an unwarranted refinement in view of the

fact that for a given value of s_1, it yields a result generally less than 10% in excess of that obtained by the formula $M = s_1 I/c$. In conventional design procedure using WF or similarly shaped members, the two limiting moments may be considered equal with the understanding that if we consider the bending moment as $M = s_1 I/c$ we err considerably on the safe side and if we use $M = 2s_1 A\bar{y}$, then we may err slightly on the unsafe side. Given a choice between the two, we prefer to err on the safe side and thus would prefer $M = s_1 I/c$. This has an added advantage in that the term I/c is readily available whereas the term $A\bar{y}$ is not as yet found in any steel handbook.

USE OF LIMIT MOMENT

The reader may well wonder why the formula for limit moment was introduced if its use is not generally recommended. The reason for doing so will become apparent when the concept of designing structures by means of assuming plastic hinges is explained but, for the present, it may be said that the reason is twofold. In the first place, as has been stated, most of the tests of actual structures were made with rectangular sections and it was found that the limiting value of moment as found from $M = 2s_1 A\bar{y}$ was extremely close to the actual experimental value whereas $M = s_1 I/c$ would have been in error by 33%. If model WF sections had been used, then the

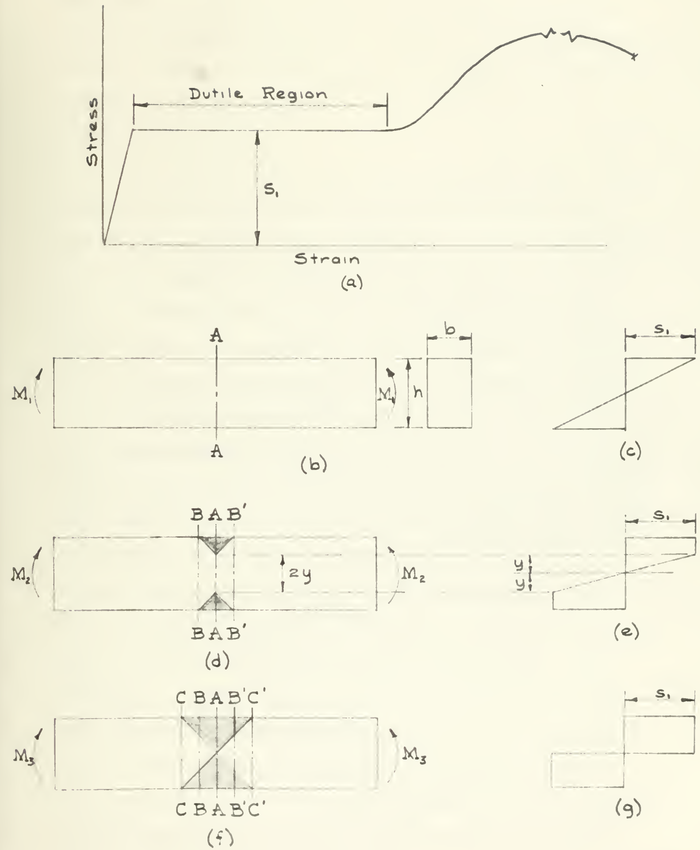
error occasioned by the use of $M = s_1I/c$ would have been only 5% to 7% but even this could hardly have been tolerated in comparing predicted with experimental results. The second reason for using the moment given by $M = 2s_1A\bar{y}$ is even more important than the first, in that it gives us a rational explanation of the action which we have chosen to call "the formation of plastic hinges". The elastic theory supplies no such explanation and, as applied to structural design, actually implies that such an action cannot occur.

PLASTIC HINGE AND PLASTIC MOMENT

To the knowledge of the writers of this thesis, the term "plastic hinge" is not to be found in any of the standard works on structural design or strength of materials. Therefore no standardized definition will be given or even attempted here. It may be recalled that when we were discussing ductility, we stated quite arbitrarily that ductility was a cause and plasticity was an effect. It may now be stated, and just as arbitrarily, that ductility is the cause and that plastic hinges are the result when a member is subjected to bending stresses in excess of the elastic limit stress. Thus, just as ductility is the property of a material by virtue of which it deforms extensively under a constant or slightly increasing stress, so plastic hinges are a property of a point or points in a structure by virtue

of which excessive bending occurs at the point or points under a constant or slightly increasing bending moment. Figure 3 is similar to figures 1 and 2 and will facilitate the explanation of the formation of plastic hinges. Figure 3(a) shows the idealized stress-strain curve for mild steel for reference. The beams of section AA shown in figure 3 have externally applied moments in the interests of simplicity although it must be assumed that the full value of the moment acts only at section AA in each beam just as if a concentrated load were applied at section AA. Consider the beam in figure 3(b) loaded with a moment M_1 which stresses the extreme fiber in the beam up to the elastic limit σ_1 figure 3(c). M_1 is then equal to $\sigma_1 bh^2/6$. If the elastic limit in the top and bottom fibers of the beam is exceeded by applying a moment M_2 (figure 3(d)), then the stress distribution over the cross section of the beam becomes as shown in figure 3(e) and the magnitude of M_2 is expressed as $M_2 = \sigma_1 b(h^2/4 - y^2/3)$. When moment M_3 (figure 3(f)) is applied so that all the fibers over the entire cross section have become ductile (figure 3(g)) then y in the previous equation equals zero and we obtain $M_3 = \sigma_1 bh^2/4$ which is merely the formula $M = Z\sigma_1$ for a rectangular section. Since all fibers at section AA in the beam shown in figure 3(f) are now in the ductile region of the stress-strain curve in figure 3(a), the beam still

FIGURE 3


Dutile Region
Stress
Strain
S_1
(a)
A
A
M_1
M_1
b
h
(b)
s_1
(c)
B A B'
B A B'
M_2
M_2
2y
(d)
s_1
y
y
(e)
C B A B' C'
C B A B' C'
M_3
M_3
(f)
s_1
(g)

bend extensively at section AA under constant moment (M_p) or a slightly increasing moment. Thus section AA may be called a "plastic hinge" and moment M_p may be called the plastic moment for the section.

EXTENT OF DUCTILE FLOW

The shaded portions of figures 3(d) and 3(f) represent the area of the beam in which ductile flow has occurred. There is no apparent means of predicting the exact extent of this area but the experiments run in connection with this thesis verified the assumption that the area of ductile flow is very limited in extent. Beyond the area of ductile flow, normal elastic behavior exists in every respect. Thus at sections BB and B'B' in figure 3(d), the stress pattern is as shown in figure 3(c). Similarly, at sections CC and C'C' in figure 3(f), the stress pattern is also as shown in figure 3(c) while at sections DD and D'D' in figure 3(f), the stress pattern is as shown in figure 3(e). The area of depth 2y, as shown in figure 3(d), between the zones of ductile flow, is referred to herein as the elastic core. The existence of this elastic core means that the beam will act elastically under any moment less than the plastic moment. This means that ordinary elastic deflection formulas may be used even after ductile flow has occurred (as in figures 3(d) and 3(e)) and that the plot of load versus deflection for a simple beam under a single

concentrated load should be of constant slope right out to the point of failure. This action of a simple beam under an increasing concentrated load is discussed more fully in the section on experimental results.

REDUNDANT STRUCTURES

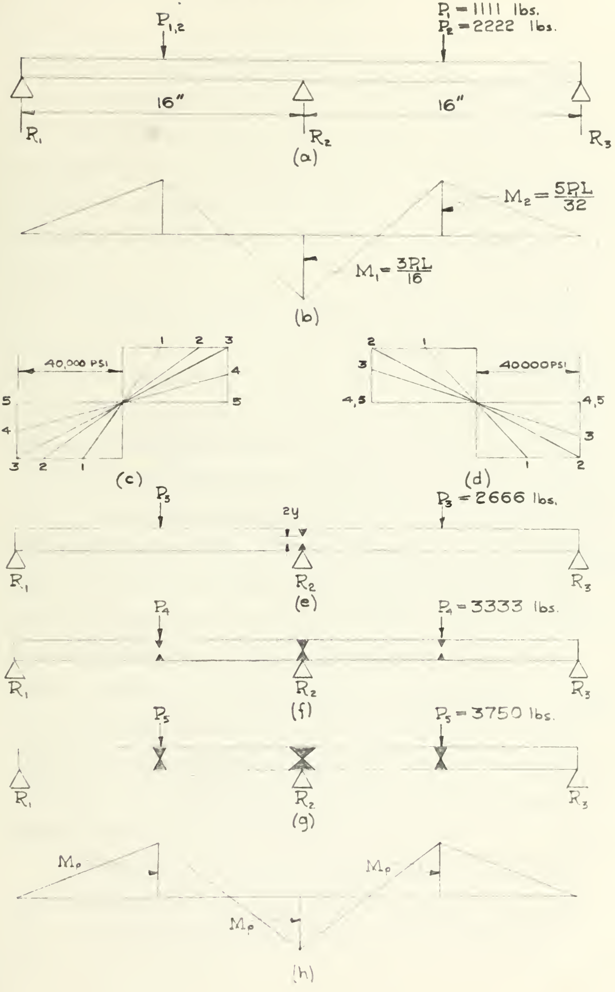
It was seen in Figure 3, and the text which accompanied it, that in the case of a simple beam with a concentrated load at the midpoint, the formula $M = 2s_1A\bar{y}$ gives a limiting value of the load P which is 50% in excess of the value given by the formula $M = s_1I/c$ when the cross section of the beam is rectangular. It has also been pointed out that when the beam is of the more common WF or I section, the formula $M = 2s_1A\bar{y}$ gives a value of the load P which is generally less than 10% in excess of that given by $M = s_1I/c$. By ignoring this slight difference, the structural engineer has erred on the safe side when designing simple beams but usually without knowing that he had done so. In redundant beams conditions are strikingly different. The essence of redundancy is that two or more members, reactions, or restraints function to the same end. Elastic criteria prescribe the relationship between the carrying capacity of component members of a given structure. These elastic criteria are the basis of the present accepted method of designing indeterminate structures. The method of assumed plastic hinges stimulates

another relationship between the carrying capacity of component members of a given structure. The verification of this relationship is the purpose of this thesis.

ACTION OF A CONTINUOUS BEAM

We shall now offer a theoretical analysis of the functioning of a two span continuous beam under continuously increasing loads. This theoretical analysis is verified by experimental test results which will be discussed later. Let us consider the beam shown in figure 4. For the purposes of simplifying the analysis, we have chosen a beam of rectangular cross section bd with two equal spans each loaded with an equal concentrated load P at its midpoint. For the sake of getting comparative numerical values of the loads P under various loading criteria, let us assume that the value of the yield stress s_1 for the beam in question is equal to 40,000 psi and that a safety factor of two is desired. Thus when using elastic criteria, a working stress of 20,000 psi will be used which is in line with current practice. However, when we apply the safety factor to critical loads as obtained by the method of assumed plastic hinges, we shall use the full yield strength of the material and then divide the failure load by the factor of safety. In the days when the working-stress idea was developed by dividing the elastic-limit stress by a factor of safety, one might have arrived at identical

FIGURE 4



$P_{1,2}$
$P_1 = 1111$ lbs.
$P_2 = 2222$ lbs.
16"
16"
R_1
R_2
R_3
(a)
$M_2 = \frac{5P_1L}{32}$
$M_1 = \frac{3P_1L}{16}$
(b)
40,000 PSI
1 2 3
4
5
5
4
3 2 1
(c)
2 1
3
4,5
40000 PSI
4,5
3
1 2
(d)
P_3
$P_3 = 2666$ lbs.
2y
R_1
R_2
R_3
(e)
P_4
$P_4 = 3333$ lbs.
R_1
R_2
R_3
(f)
P_5
$P_5 = 3750$ lbs.
R_1
R_2
R_3
(g)
M_p
M_p
M_p
(h)

results if, however, the safe load had been multiplied by the factor of safety and the working stress defined as equal to the elastic limit stress. It seems too bad that this was not done, as then the transition from the elasticity-strength philosophy to that of figuring the ultimate failure load would seem less abrupt.

Returning to figure 4, let us further assume that the beam is a one inch square bar and that the distance between supports is 10 inches. For this section, therefore, the value of I/c is equal to $bh^2/6$ or 0.167 while the value of $2A\bar{y}$ is equal to $bh^2/4$ or 0.25. Then the maximum permissible moment under the elastic working stress theory is equal to I/c times the working stress or 0.167 (20,000) which is 3333 in-lb with the understanding that a safety factor of two against exceeding the elastic limit stress has been provided. The value of the plastic moment is equal to the yield stress times $2A\bar{y}$ or 0.25(40,000) which is 10,000 in-lb with the understanding that the safety factor of two will be applied not to the working stress, but to the load which causes failure. With these figures in mind, we can proceed with the analysis.

Let P_1, P_2, P_3, P_4, and P_5 represent continuously increasing loads on the beam shown in figure 4 and let 1, 2, 3, 4, and 5 of figures 4(c) and 4(d) represent

the stress patterns on sections under loads P and over reaction R_2 respectively. For load P_1, within the elastic range, the bending moment diagram is shown in figure 4(b). Since elastic criteria require that the allowable working stress must not be exceeded at any point in the beam, we take the largest moment, $3PL/16$, and set it equal to the value of maximum elastic resisting moment, 3333 in-lb, and solve for P_1. P_1 for the given beam is then found to be equal to 1111 lb. And this is the maximum load permitted under elastic criteria. Since the moment under the loads is equal to 5/6 of the moment over R_2, the stress pattern under the load P_1 is as shown in figure 4(c)1 while the stress pattern at R_2 is shown in figure 4(d)1. Doubling the loads P_1 so that loads P_2 are equal to 2222 lbs. each will not alter the shape of the moment diagram of figure 4(b). It will merely double the values of the moments and hence double the value of the extreme fiber stresses as shown in figures 4(c)2 and 4(d)2 to 33,333 psi and 40,000 psi respectively. Since the yield stress has been reached in the extreme fibers at R_2, any load greater than P_2 will cause ductile flow to take place in the region near R_2 just as occurred under the load in the case of the simple beam in figure 3 under loads P_2 and P_3. However, the continuity of the beam tells us that as long as there is a continuous elastic core throughout

the length of the beam, then the moment diagram of figure 4(b) still applies and the deflections are of the normal order of magnitude. Thus if we take load P_3 as the load which will cause the elastic limit stress to be reached under the loads, then the resulting stress diagrams will be as shown in figures 4(c)3 and 4(d)3. The load P_3 may be solved for and is found to be equal to 2666 lbs. The shaded area of figure 4(e) shows the portion of the beam which has been stressed beyond the elastic limit. The depth of the elastic core, $2y$, if desired, could be obtained from the relationship $M = s_1 b(h^2/4 - y^2/3)$. The depth of this core is unimportant so long as we are satisfied that it exists. Its existence tells us, as previously stated, that the entire beam is still behaving elastically.

Now let us apply load P_4 of such magnitude that all of the fibers in the section over R_2 are stressed to the ductile region of the stress-strain curve. In other words, P_4 is a load just sufficient to form a plastic hinge at R_2. Figure 4(f) shows the areas which are now stressed beyond the elastic limit and since the elastic core is now no longer continuous throughout the beam, the moment diagram of figure 4(b) no longer applies. However, the elastic core is continuous on either side of R_2 so complete failure still cannot take place. Since a plastic hinge has now been developed at R_2, the moment

in the beam at R_2 will now remain constant and equal to the plastic moment, Mp = 10,000 in-lbs. In effect, what we now have is two simple beams with a stiffening couple of 10,000 in-lbs applied at their adjacent ends.

In theory, any load slightly less than P_4 will permit the assumption of an elastic curve throughout the length of the beam. Thus any load a differential amount less than P_4 would still produce a moment diagram like figure 4(b) and thus we may solve for P_4 by equating Mp = 10,000 in-lb to $3P_4L/16$ and find that P_4 is equal to 3333 lbs. As P_4 is reached, the plastic hinge at R_2 is formed and yields just as a tensile specimen yields when the yield point is reached. This yielding causes a redistribution of moments and the point of inflection moves nearer to R_2. Professor Van Den Broek, in his book, <u>Theory of Limit Design</u>, (see bibliography #1, pages 50-51) had predicted that the yielding over R_2 in going from loading P_3 to P_4 would be gradual but our experimental results (to be discussed more fully later) showed conclusively that P_4 is a well defined and predictable load and that loads just slightly less than P_4 show no tendency to cause deflections of greater magnitude than ordinary elastic deflections.

Load P_5 is the load which is just sufficient to cause plastic hinges to form under the loads. As load P_5 is reached, there is no longer a continuous elastic

case on either side of R_2 as in figure 4(f) but instead, we now have four elastic segments connected by three plastic hinges as shown in figure 4(g). The beam no longer behaves elastically but, instead, acts as an articulated structure. Even after complete failure, our experiments showed that the segments between the plastic hinges were perfectly elastic in their behavior and that the extent of the region of ductile flow was extremely limited. This behavior is what led us to the term plastic hinges to describe the action of ultimate failure of a point in a beam. Under load F_5, the moments under the loads and over R_2 are all equal to each other and the final moment diagram is as shown in figure 4(h). Note that the point of inflection is now midway between the loads and R_2. The manner of solving for F_5 will be discussed in detail a little later but it is sufficient here to note that F_5 is equal to 3,750. Since this is the value of the load F required to produce failure, we divide it by the desired safety factor to determine the allowable working load F. Since we decided at the beginning of this discussion to use a safety factor of two for both elastic criteria and failure criteria, the working load which gives a safety factor of two against failure is 1,875 lbs. This is 50% higher than the load F_1 which the elastic theory tells us has a safety factor of two also.

FAILURE IN A STRUCTURE

Now that we have demonstrated, in theory at least, the exact actions that take place in the continuous beam of figure 4 up to and including complete failure, the reader may well wonder how we propose to solve for the load which will cause complete failure and whether such a solution is more or less difficult than that performed under elastic criteria. Before going into the proposed method of solution, however, it may be a good idea to say a few things about failure. Failure in structural engineering is not a question of rupture. A steel beam can be ruptured only if we saw it, shear it, or burn it in two. If a structure fails, its beams will be bent and twisted but they will not rupture. Rupture, when it occurs, generally takes place after failure as the consequence of an inadequately designed connection detail. Therefore, when we say that we intend to solve for the failure load, we mean that load which causes uncontrolled deformations to take place. Thus in figure 4, we saw that even when a plastic hinge had formed over the middle support, uncontrolled deformations could not occur since an elastic core extended from reaction to reaction. Only under load P_5, when all three plastic hinges had formed, could uncontrolled deformations occur. And so we call P_5 the failure load, PF, and divide it by the safety factor to get the working load.

Referring to Figure 4 again, we see that if the safety factor is two, then the working load is 1,[illegible]75 lbs. and since this is less than P_2 ([illegible] lbs.), the load which first caused the elastic limit stress to be reached, we conclude that under the working load we wouldn't reach a condition of ductile flow anyway. But if the safety factor were 1.5, then the working load would be [illegible] lbs. and since this is between P_2 and P_3, we would conclude that the extreme fibers over [illegible] would be stressed beyond the elastic limit. Traditionally, we are never supposed to countenance exceeding the elastic limit but why frown on it when we continually exceed the elastic-limit stress as we hammer out dents and cold-straighten beams before fabrication.

FACTOR OF SAFETY

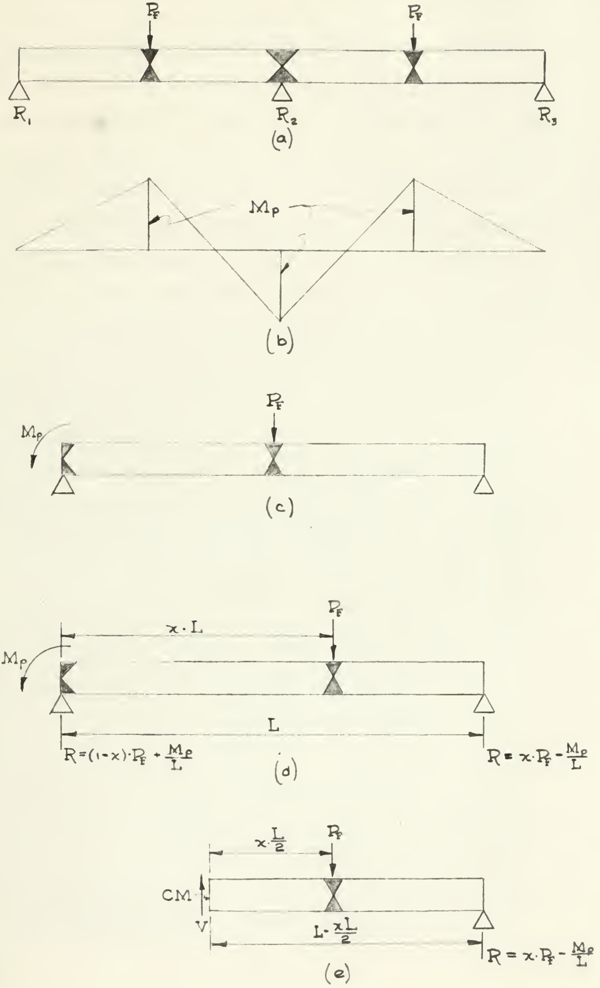
It may be argued that we have taken liberties with the factor of safety. This has not been our intention. No one alone can decide on the proper factor of safety. We have merely tried to illustrate a method. This method is as independent of the assumed factor of safety as it is independent of the elastic-limit stress which is also assumed to be known. The concept of assumed plastic hinges offers a means whereby a constant factor of safety may be selected. This factor should come closer to providing a measure of the margin of safety against failure in a structure than does the one now applied

and by means of which we decide working stresses. Under the working-stress theory prescribed by elastic criteria, the designer generally assumes that he has employed a safety factor 1 to his structure equal to the value of the elastic limit stress divided by the assumed working stress. Since the elastic limit stress of mild steel is generally taken as being between 36,000 and 40,000 psi and since most codes prescribe working stresses of 18,000 or 20,000 psi, the engineer assumes that this safety factor is numerically equal to two. The factor of safety against exceeding the elastic limit stress may be two but only in the case of a simple beam is the factor of safety against failure equal to two. Even the skeptical reader will have to admit that it appears that in the case of redundant beams and frames, the factor of safety against failure is actually in excess of two if conventional methods of elastic working stress design are used. In applying the concept of plastic hinges to the design of redundant structures, we propose to multiply the working load by a factor of safety and to design the structure so as to fail under this load. We will now develop the method of solving for this failure load.

GENERAL SOLUTION FOR FAILURE LOADS

Figure 5(a) shows a beam similar to that shown in figure 4. Let P_f represent the loads which are just

FIGURE 5

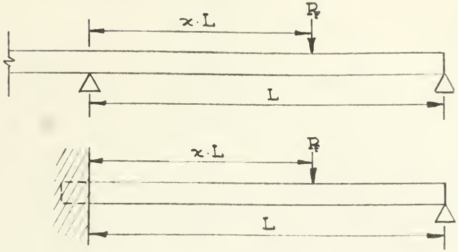


P_F
R_1
R_2
R_3
(a)
M_p
(b)
P_F
M_p
(c)
$x \cdot L$
P_F
M_p
L
$R = (1-x) \cdot P_F + \frac{M_p}{L}$
$R = x \cdot P_F - \frac{M_p}{L}$
(d)
$x \cdot \frac{L}{2}$
P_F
CM
V
$L - \frac{xL}{2}$
$R = x \cdot P_F - \frac{M_p}{L}$
(e)

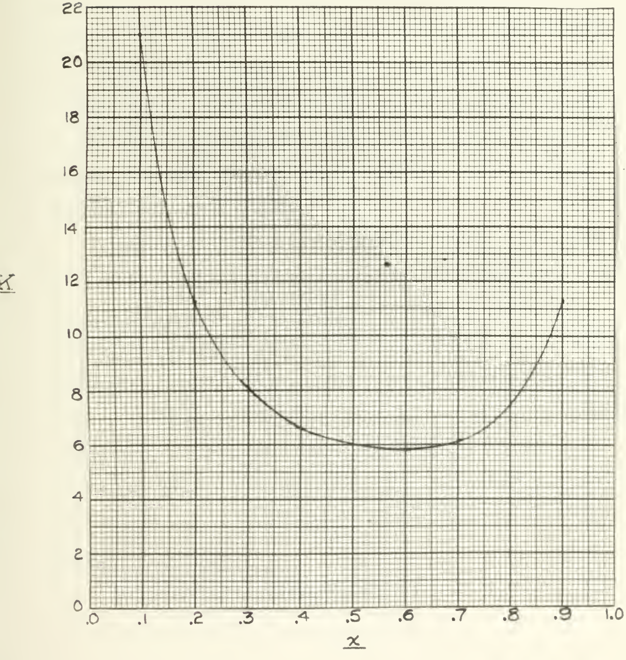
sufficient to cause failure and let M_p be the plastic moment which may be solved from $M_p = \sigma_y Z$ as previously shown. As before, the shaded areas on the beam represent the area in which plastic flow has occurred. Figure 5(b) shows the moment diagram for the beam under the failure loads. Note that the inflection point is midway between the two plastic hinges on either side of the middle reaction. Figure 5(c) shows the half of the beam to the right of R_2 taken out as a free body with the effect of the portion of the beam to the left of R_2 shown as an externally applied moment M_p. Figure 5(d) shows the general case of figure 5(c) where x is a decimal part of the length of the span. From the equations of equilibrium, the reactions are solved for and are shown in terms of P_f, M_p, x, and L in figure 5(d). We have seen in figure 5(b) that, under failure conditions, the point of inflection or zero moment must lie midway between the plastic hinges. Hence we may take out the free body of figure 5(e) by cutting the beam at the point of inflection and replacing the effect of the rest of the beam by the shear V. Since the sum of the moments about the point of inflection must equal zero, we may equate the moments of figure 5(e) about O.P. to zero and say: $(xL/2)P_f-(L-xL/2)(xP_f-M_p/L) = 0$. Solving for P_f gives us:

$$P_f = (M_p/L)(2-x/x-x^2) \text{ or } P_f = K(M_p/L)$$

FIGURE 6



$$P_f = K\left(\frac{M_p}{L}\right)$$



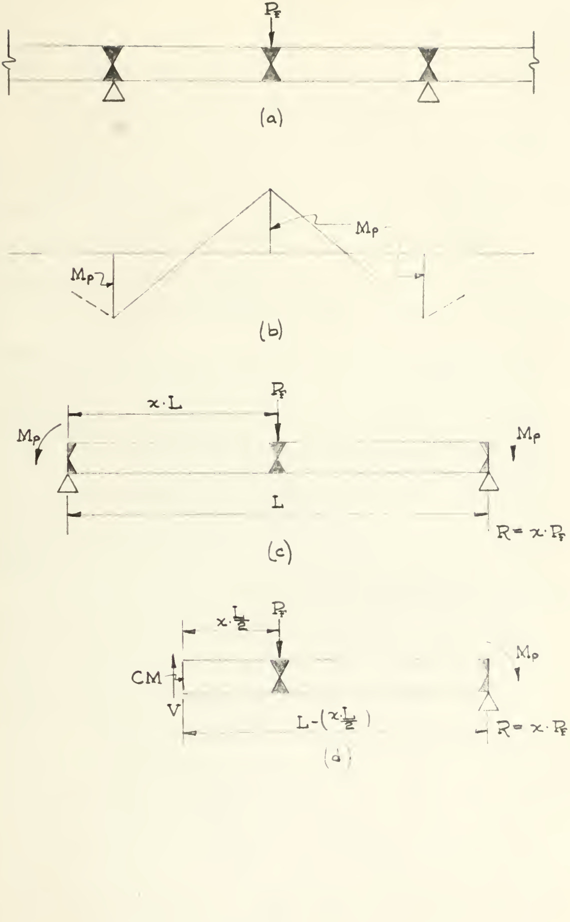
LOAD FACTOR

Since the factor K is purely a function of the position of the load on the beam and does not depend on the size or length of the beam, we may plot a curve of K vs. x. Such a curve is shown in figure 6. With this curve, and knowing the length (L) and ultimate resisting moment of the beam (M_p), it is possible to get the failure load for any position on the beam by simple multiplication. Note that figure 6 applies equally well to the exterior span of a continuous beam or to a beam which is built-in at one end and simply supported at the other. The value of P_f, found by multiplying K times the plastic moment of the beam divided by its length, is then divided by the desired safety factor to find the working load.

Figure 7(a) shows the interior span of a continuous beam under failure load P_f. Figure 7(b) represents the moment diagram at the instant of complete failure. Note that regardless of the position of P_f on the beam, all three plastic hinges must form before failure can occur. Following the same reasoning as in the preceding development, we may take out successive free bodies as shown in figures 7(c) and 7(d) and, since the sum of the sum of the moments at R.H. in figure 7(d) is equal to zero, we may say:

$$P_f(xL/2) + M_p - (L-xL/2)xP_f = 0$$

FIGURE 7



P_F
(a)
M_p
M_p
(b)
$x \cdot L$
P_F
M_p
M_p
L
$R = x \cdot P_F$
(c)
$x \cdot \frac{L}{2}$
P_F
CM
M_p
V
$L - (x \cdot \frac{L}{2})$
$R = x \cdot P_F$
(d)

and by solving for F_f we have:

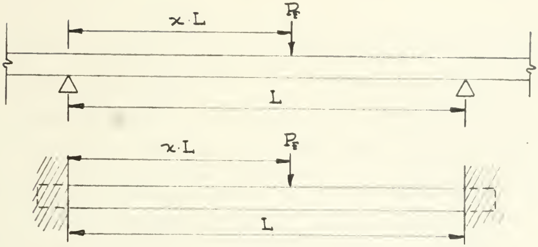
$F_f = (M_p/L)(12/m-x^2)$ or $F_f = K_1(M_p/L)$.

Figure 8 shows the plot of K_1 vs. F_f and is used in exactly the same manner as Figure 6. Note that figure 8 applies equally well to interior spans of continuous beams and to beams which have both ends built-in. Figure 8 must be used with some caution in the case of built-in beams, however, since most authorities are agreed that a perfect built-in condition is unattainable either in practice or in the laboratory. The equivalent of the ideal built-in condition can be obtained by taking advantage of symmetry and this was the method employed in connection with this thesis for checking the theoretical predictions.

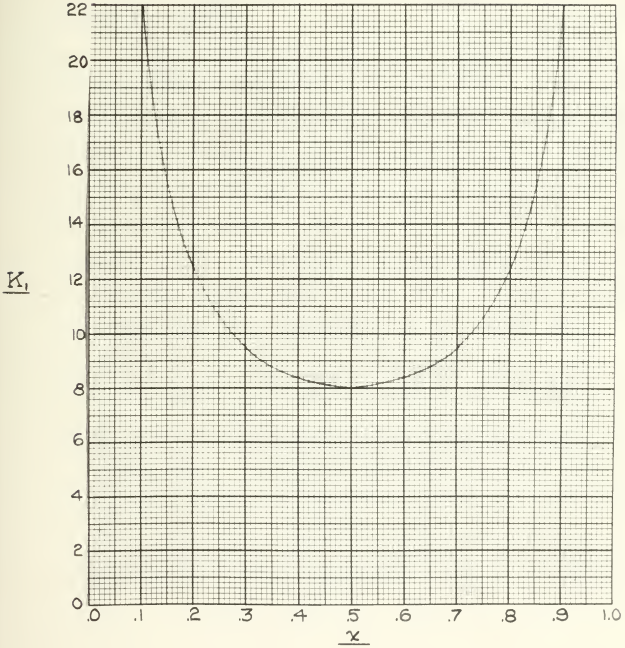
ADVANTAGES OF ASSUMING PLASTIC HINGES

At first glance, the problem of predicting failure loads for a structure seemed to call for a detailed investigation of the elastic-plastic behavior of the structure under gradually increasing loads. This laborious investigation is not necessary, however, if only the safety factor against collapse is required. It is this fact, in addition to the fact that a true factor of safety against collapse is found, that makes the theory of assumed plastic hinges so attractive. It may be recalled that the formulas which formed the basis for figures 6 and 8 were derived purely from the formulas

FIGURE 8



$P_F = K_1\left(\frac{M_P}{L}\right)$

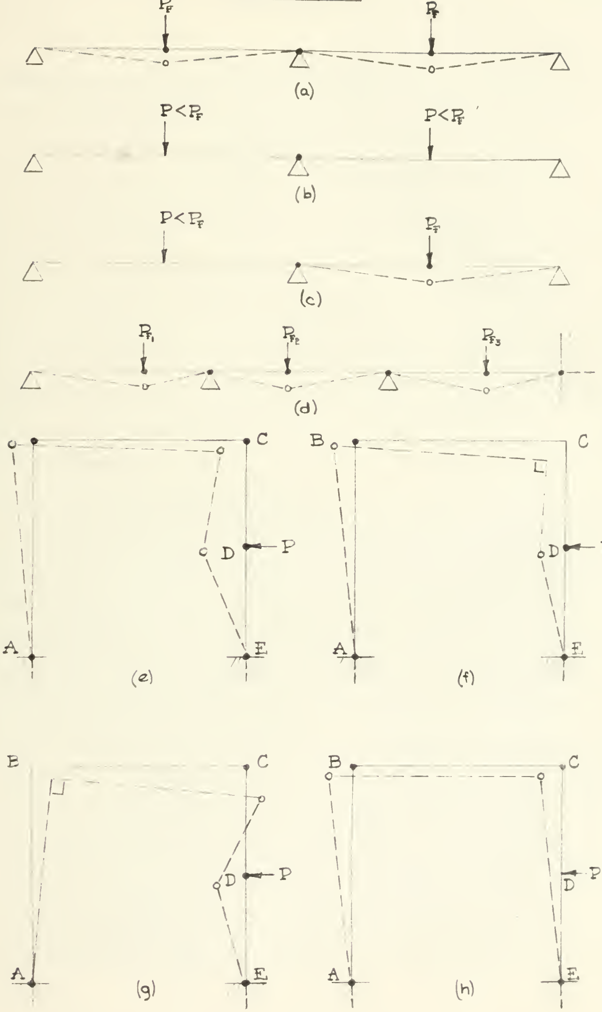


of statics without the aid of any supplementary formulas in spite of the fact that the structures considered are generally regarded as indeterminate. The reason that this could be done was that the assumed plastic hinges reduced the degree of redundancy of the structure. If a sufficient number of plastic hinges are allowed to form (or assumed to be allowed to form), then the structure will be transformed into a mechanism capable of solution by the formulas of statics.

MECHANISM DEFINED

To test whether this stage has been reached, it is sufficient to replace all assumed plastic hinges by perfect hinges and consider the structural members which are joined by these hinges as completely rigid. If the structure, modified in this manner, is capable of at least an infinitesimal deformation despite the assumed rigidity of its members, then the actual formation of the assumed plastic hinges would lead to collapse of the structure (the term "mechanism" as used herein after bibliography ([illegible], pages 1-[illegible]). For instance, the beam shown in figure 5(a) becomes a mechanism if plastic hinges develop at the three cross sections indicated by the heavy dots. Indeed, if these plastic hinges were replaced by perfect hinges and the members between these hinges considered as rigid, the beam becomes a mechanism capable of the type of deformation which is indicated in

FIGURE 9



P_F
P_F
(a)
$P < P_F$
$P < P_F$
(b)
$P < P_F$
P_F
(c)
P_{F_1}
P_{F_2}
P_{F_3}
(d)
B
C
D
P
A
E
(e)
B
C
D
P
A
E
(f)
B
C
D
P
A
E
(g)
B
C
D
P
A
E
(h)

dotted lines in Figure 9(a). In Figure 9(b), only the first plastic hinge to form has been shown and it should be apparent that no mechanism is formed by replacing this one plastic hinge with a perfect hinge as the result would merely be a pair of rigid simple beams connected by a hinge. In the example given in Figure 9(a), the actual plastic hinges were so arranged that the entire structure was transformed into a mechanism. In other cases, however, the actual plastic hinges may transform only part of the structure into a mechanism. These two types of collapse may be called total and local collapse.

COLLAPSE OF CONTINUOUS BEAMS

For continuous beams, collapse is very often local. For instance, the beam shown in Figure 9(c) has suffered local collapse in the right span under the action of P_F but no collapse in the left span under the action of a load P which is less than P_F. Thus if the indicated plastic hinges in Figure 9(c) were replaced by perfect hinges and the members considered as rigid, then the right span would become a mechanism capable of deformation as indicated by the dotted lines while the left span would become merely a rigid simple beam. The fact that local collapse can occur in continuous beams makes possible the independent application of Figures 4 and 8 to continuous beams involving both types of joints.

For instance, figure 9(d) shows a continuous beam for which the failure loads P_{f1}, P_{f2}, and P_{f3} in each span could be predicted as follows: P_{f1} could be predicted from figure 6 and P_{f2} and P_{f3} from figure 8. It should be apparent by now that the location of assumed plastic hinges in continuous beams acted on by concentrated loads is a matter which offers very little difficulty and since the experiments run in connection with this thesis involved only concentrated loads, no discussion of the location of plastic hinges under uniformly distributed loads is presented here.

RIGID FRAMES

The problem of determining the failure loads for rigid frames by the theory of assumed plastic hinges is basically just as simple as in the case of continuous beams. That is, a sufficient number of plastic hinges are assumed to transform the structure into a mechanism and then the load which is just capable of forming these plastic hinges is solved for by the formation of equations.

PLASTIC HINGES IN RIGID FRAMES

The only difficulty encountered is that the location of the plastic hinges in just sufficient number to transform the structure into a mechanism is not quite as obvious as in the case of continuous beams. For instance, figures 9(e), 9(f), 9(g), and 9(h) show the frame ABCDE acted on by a side load P acting at B. If the member

has made a study of the subject of indeterminate structures, he will recall that the largest values of moment, as computed by elastic means, are generally located at the corners, at the junctions, or at the load points of the structure. With this fact in mind, it would perhaps seem logical to assume plastic hinges at A,B,C,D, and E in figure 9(e) and thus transform the rigid frame into a mechanism capable of the type of deformation indicated by the dotted lines. A closer inspection of the situation, however, reveals that the structure will also act as a mechanism if a plastic hinge is omitted at either C, as shown in figure 9(f), or at B, as shown in figure 9(g), or at D, as shown in figure 9(h). The omission of any more assumed plastic hinges, on the other hand, results in a structure which is incapable of becoming a mechanism when the assumed plastic hinges are replaced by perfect hinges and the members between the hinges considered rigid. In the simple frame shown, the determination of the number of plastic hinges just sufficient to create a mechanism is not too difficult but, in more complex frames, the job might be quite tedious. The following dictum is presented now to facilitate the job: "In a well designed (and especially a properly detailed) structure which is n-fold redundant, it is necessary and sufficient that (n+1) plastic hinges be formed in order that the structure

be transformed into a mechanism."

MAXIMUM NUMBER OF PLASTIC HINGES

This dictum applies to all redundant structures and was neglected when discussing continuous beams only because its application at that point was deemed contrary to the interests of simplicity. In the case of continuous beams, the number and location of the proper plastic hinges is generally quite obvious but to illustrate the application of the dictum, take a beam built in at both ends. Such a beam is twice redundant and hence the dictum requires three plastic hinges in order to transform the structure into a mechanism. This agrees with the assumptions made in arriving at figure 8.

Returning to figures 7(e) through 7(h), we find that the frame ABCDE is thrice redundant and hence four plastic hinges will be necessary to transform the structure into a mechanism. This agrees with the assumptions made in figures 7(f) through 7(h), for in each of these three possible conditions of forming a mechanism, there are four plastic hinges. It now becomes a matter of logic and test to determine which combination is the correct one. Logic tells us that since the load is tending to push the frame over on its side, final failure cannot occur until plastic hinges have formed at both A and E as shown in figures 7(f) through 7(h). It would also seem logical to assume that a plastic hinge

could form at D under the load P. Such an assumption would rule out the case shown in figure 9(h) and leave only the cases shown in figures 9(f) and 9(g). The decision as to whether to omit the plastic hinge at C as in figure 9(f) or at D as in figure 9(g) is generally a matter of test although logic may serve to make the test conclusive in itself.

ADMISSIBILITY OF ASSUMED PLASTIC HINGES

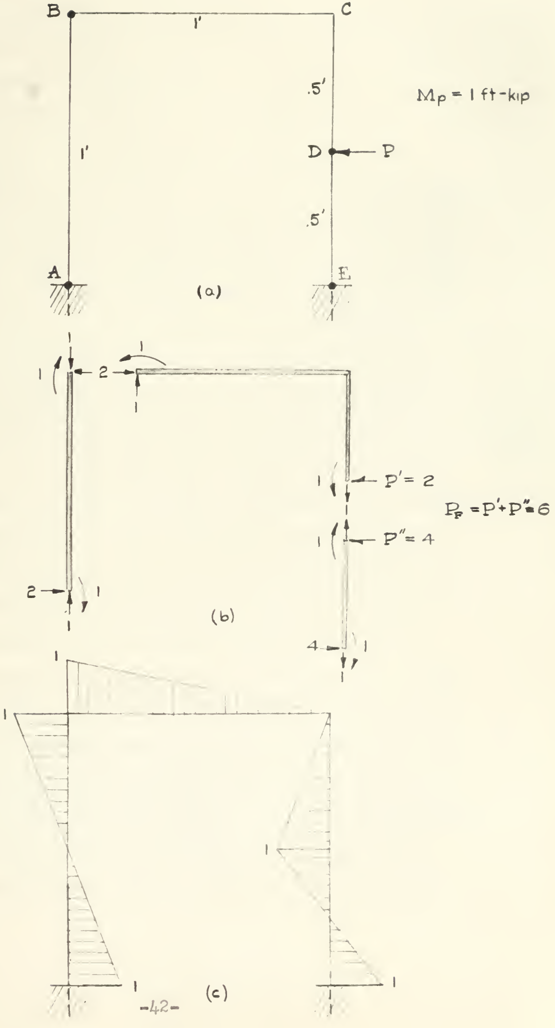
The following dictum is proposed as both a basis and a criteria for testing redundant structures for the validity of the assumed plastic hinges: "In the design of a redundant structure for failure loads, it is not necessary to use the conditions of elasticity to determine the redundants. It is only necessary to assume that the redundant moments are equal to the plastic moment of the members at the points of redundancy, provided such assumptions are compatible with the conditions of equilibrium, and to solve for the load or loads just sufficient to create the assumed values of the redundant moments. If the bending moment diagram which is computed from this load, or set of loads, and the assumed values of redundant moments is such that at no cross section does the absolute value of the bending moment exceed the value of the plastic moment of the structure at the cross section, then the bending moment diagram is said to be statically compatible with the structure and the loads

will be statically admissible." (The term "statically admissible" after Bibliography #2, page 3)

TESTING ASSUMED PLASTIC HINGES

Figure 10 illustrates the method by which the preceding dictum is applied to test a given assumption for static admissibility. Figure 10(a) shows the frame ABCDE loaded in the same manner as in figures 9(e) through 9(h) and with the same assumed plastic hinges as in figure 9(f), that is, with plastic hinges at A,C,D, and E. In accordance with the preceding dictum, we assume the values of the redundants by replacing them with the values of the plastic moment. For the sake of simplicity, we have assumed a value of 1 ft-kip for the plastic moment and a value of 1 foot for the length of the members of the frame. In order to replace the redundants with assumed values and insure that the conditions of equilibrium are not violated, we have taken each section between plastic hinges out as a free body as shown in figure 10(b). Note that only the values of the redundant moments are assumed and that the values of shearing and axial stresses at each section are solved for by the formulas of statics. All stresses at the joints are shown as applied externally to the free bodies in figure 10(b) and the numerical values shown are correct for the assumed value of plastic moment and member lengths. Since it can easily be seen that all the free bodies are

FIGURE 10



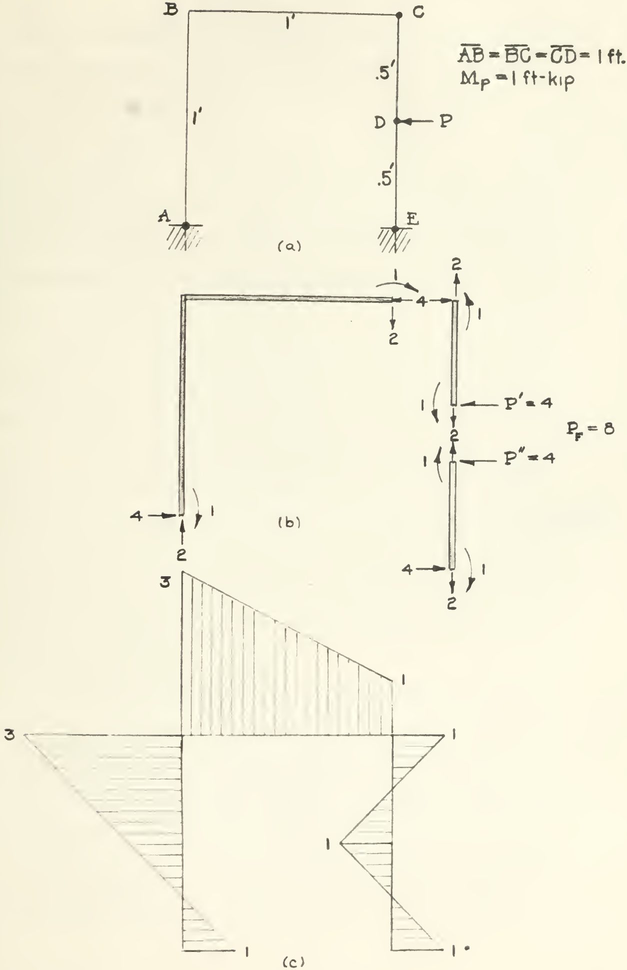
statically admissible it now is necessary to check the structure as a whole for static admissibility to insure compliance with the dictum cited in the preceding paragraph. It is easy to see in the example given that the structure as a whole is statically admissible since $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. The final check on static admissibility is to plot the bending moment diagram to see that the elastic or limiting moment is not exceeded at any point. Since we have already fixed the value of the bending moment at points A,B,D, and E as equal to 1 ft-kip, it is only necessary to solve for the moment at point C in order to have sufficient data to plot a complete bending moment diagram. By solving either member BC or CD as a free body, we find that the value of the moment at C is 0 and with this final information, we plot the bending moment diagram as shown in figure 11(c). Since the assumed moments and computed load P_f have already been shown to be externally statically admissible and since the bending moment diagram has shown the same moments and load to be internally statically admissible, we have satisfied the conditions of the dictum and we may say that the failure load is equal to 6 kips. Note that we are not concerned with the order in which the plastic hinges appear. It is only necessary to realize that until the (n+1)th plastic hinge does appear, the structure will not act as a mechanism and

hence deflections will be of the order of elastic deflections.

INCORRECT ASSUMPTION

Now let us take the same frame and make the assumption of plastic hinges shown in figure 11(a). The plastic hinges at A, B, and D remain the same as in figure 10(a) but the fourth hinge has been assumed to form at C instead of at E. We will now again apply the criteria proposed by the general theorem to see if these criteria will rule out incorrect assumptions. As before, we assume that the redundant moments are equal to the plastic moments and take the free bodies between all plastic moments as shown in figure 11(b). Solution of these free bodies by the formulas of statics gives us the values of redundant reactions shown in figure 11(b) and a check of the free bodies and of the structure as a whole, shows that the loads and redundant reactions are externally statically admissible. It is now necessary to check the second criteria of the theorem and see that the plastic moment is not exceeded at any point in the structure. As in the example of figure 10, all of the data for constructing a bending moment diagram is available as assumed except for the value of the moment at E. A solution of either AE or EC as a free body shows that the moment at E under the action of the loads and redundants shown is equal to 5 ft-kips. Although such

FIGURE 11



B
C
1'
$\overline{AB} = \overline{BC} = \overline{CD} = 1$ ft.
$M_p = 1$ ft-kip
.5'
1'
D
P
.5'
A
E
(a)
2
1
4
1
2
1
P' = 4
2
$P_F = 8$
1
P'' = 4
4
1
(b)
2
4
1
2
3
1
3
1
1
1
1
(c)

a moment is obviously impossible in a structure whose plastic moment is only 1 ft-kip, the bending moment diagram is sketched in Figure 11(c) to show how it violates the criteria proposed by the author. Since the bending moment obviously does exceed the plastic moment in figure 11(c), we say that the assumed plastic hinges are statically incompatible with the structure and that 6 kips is not a valid value of the failure load.

EASE OF APPLICATION

Figures 10 and 11 should have demonstrated to the reader not only the validity of the proposed theorem but also the ease of its application. Not only has the value of the actual failure load been determined but it was determined without the aid of any complementary equations or methods borrowed from the field of elasticity. Furthermore, it is now possible to apply any desired safety factor and know that in so doing, we have established a definite factor of safety against failure.

RESTRICTIONS BASED ON THEORY

It is our hope that this section on theory has given the reader a clear and rational understanding of the various phenomena which we have chosen to call "the formation of plastic hinges in simple structures". In attempting to make this explanation as rational as possible, we have tried to develop all of our concepts from facts which are generally known and accepted.

Since we have found that the basic ideas of limit design are not generally known and hence cannot be generally accepted, we have felt duty bound to our readers to offer a theoretical discussion sufficiently detailed to make a complete understanding of our work possible without reference to other works on the ductile behavior of beams and frames. In reading such a detailed explanation, however, the reader may tend to lose sight of the basic concepts which must, of necessity, be presented in the midst of much supplementary and explanatory material. We therefore propose to summarize this section by pointing out the things which our study of the subject of plastic hinge formation had led us to predict would happen. It may seem a bit presumptuous on our part to set down as predictions those things which we have since verified by experiment but such a course seems justified in this case in order to give proper credit to Professor Brathen, who made us ductility and plastic-hinge conscious and who directed our work so well toward the establishment and verification of these predictions concerning the formation of plastic hinges. We therefore feel justified in saying that these are the things which we had predicted and which formed the very basis for our experimental work:

That the ultimate moment resisting capacity of a beam made from a ductile material such as

mild steel can be accurately determined by assuming a rectangular stress distribution and using $M = 2s_1\bar{Q}$.

That this ultimate resisting moment can be accurately predicted even when the shape of the section is such that the greatest mass is concentrated at or near the elastic core (as in a circular section).

That for rectangular sections, the ultimate resisting moment will be exactly 50% in excess of the elastic-limit resisting moment as computed from $M = s_1 I/c$

That until the complete rectangular stress distribution has developed, the elastic core, however small, will cause the normal relationship between loads and deflections to be maintained.

That when a plastic hinge does form, the moment resistance will remain essentially constant as yielding occurs.

That the extent of the plastic hinge along the axis of the member will be very limited.

That the parts of the structure between the plastic hinges will exhibit perfectly normal elastic behavior.

That the yielding which occurs when a plastic hinge forms will automatically cause the most

favorable redistribution of bending moments.

That in an n-fold redundant structure, (n+1) plastic hinges must form before the entire structure is transformed into a mechanism.

That the load or loads just sufficient to cause complete failure (that is, to transform the structure into a mechanism) can be accurately predicted by the equations of statics.

That individual spans of continuous beams will act independently of one another and that failure loads for any combination of these may be predicted from the curves in figures 4 and 5.

That the advantages of assuming plastic hinges become greater as the degree of redundancy of the structure increases.

That, for a given factor of safety, the deflections occurring in a redundant structure designed against ultimate failure by the method of assumed plastic hinges will generally be less than those occurring in a similar but non-redundant structure designed by elastic methods.

DESIGN OF APPARATUS

After working out the theory involved in the previous discussion and exposing it to the test of practice, it was decided to conduct actual tests on models of simple structures to verify by experimental work the indicated theoretical results. Some difficulty was encountered at first because of the lack of a simple load tester on the campus. The design of a loader was undertaken and carried out with the idea of making it as versatile as possible, but primarily to accomplish the purposes of the test-work within the scope of this problem.

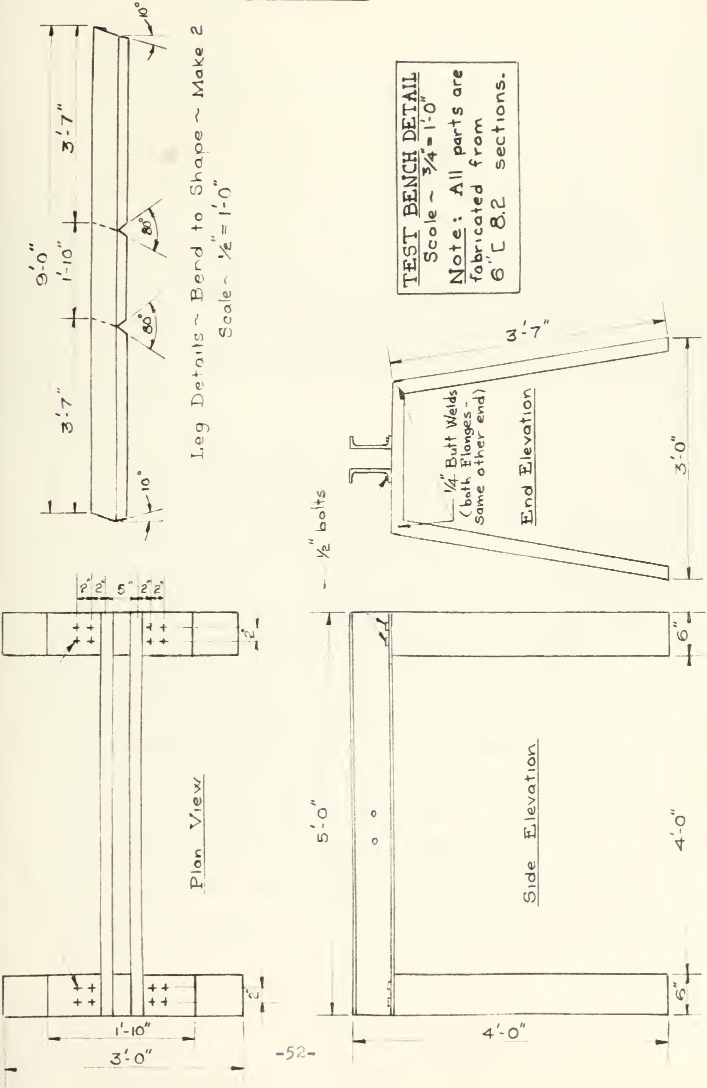
It might be well to explain the reason for designing static loading rather than availing ourselves of one of the loading machines available in the Mechanics Department Testing Laboratory. These loaders are essentially strain loaders and they apply load to give a certain rate of strain. In order to observe the phenomena which we hoped to show up in our testing this type of load application would not be satisfactory. The most important point in our tests as far as observation and measuring went, was the yield point in the beam. That is, when the plastic hinge was formed at any section of the beam. At this point in the loading there was a very definite yield of the beam, as may be seen in Figure 13, which shows the experimental results of the load test on the two span beam. If a strain loader had been used, because of the

nature of the loading, it would have released the load as this yielding of the beam, caused by the formation of a plastic hinge, occurred. Thus, the follow-through behavior after the formation of the plastic hinge in the beam could not have been observed. With a gravity loader, we were able to take advantage of the fact that the load remained on the structure even though there was sudden deflection as the plastic hinge formed. This enabled us to observe whether or not the beam stiffened after the first plastic hinge formed. This also enabled us to obtain a more accurate recording of the relationship of deflection to the applied load and in turn gave us more accurate results.

TEST BENCH

The main part of the apparatus was the test bench which was of very sturdy design in order to ensure that there would be no deflection of the bench itself caused by the loading of the models. The bench was made, as the accompanying Figure 12 shows, of six inch channels. Two nine foot lengths were bent to form the legs of the bench and two five foot lengths bolted on to the flat tops of these supports to form the cross-members for support of the knife edges which were used to support the beam models. The cross-member channels were placed with flanges out so they could be utilized for clamping on the deflection gauges and the apparatus for testing the rigid frames later on in the experimental work.

FIGURE 12



TEST BENCH DETAIL
Scale ~ 3/4" = 1'-0"
Note: All parts are fabricated from 6" [8.2 sections.
Leg Details ~ Bend to Shape ~ Make 2
Scale ~ 1/2" = 1'-0"
9'-0"
3'-7"
1'-10"
3'-7"
10°
80°
80°
10°
3'-7"
1/4" Butt Welds (both Flanges - Same other end)
End Elevation
3'-0"
1/2" bolts
Drilled for 1/2" bolts
2" 2" 5" 2" 2"
2"
Plan View
1'-10"
3'-0"
5'-0"
Side Elevation
4'-0"
6"
6"
4'-0"

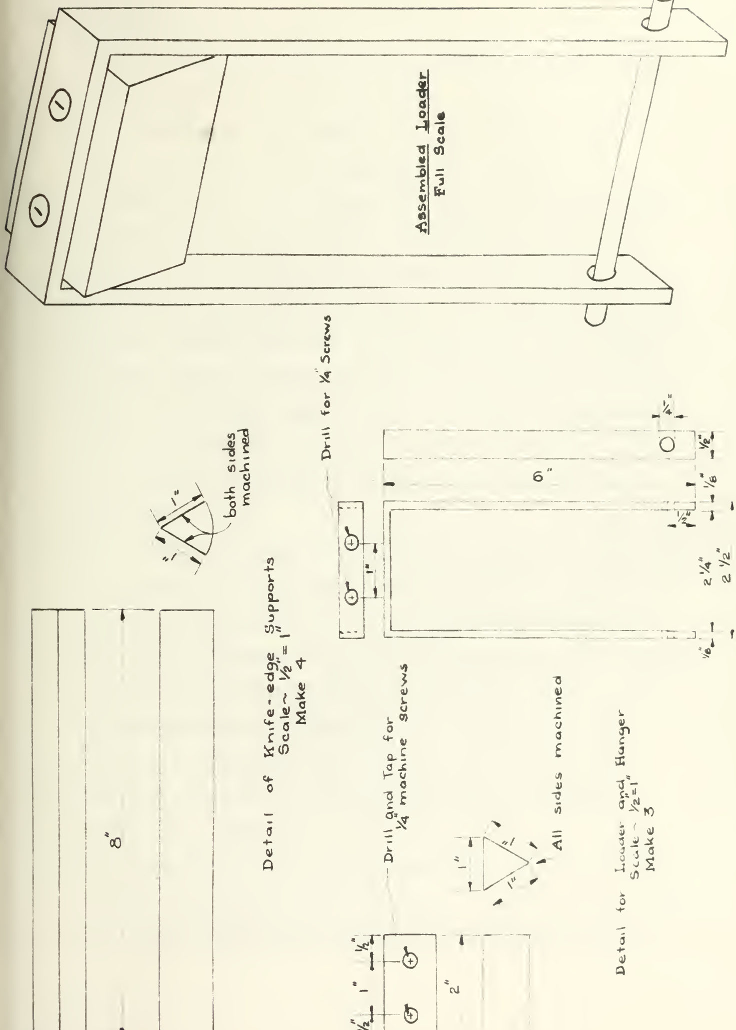
KNIFE-EDGE SUPPORTS

The knife-edges used to support the beam models were machined down to form a sharp knife-edge support. The bottoms were rounded in order to allow the support to rotate as the loads were applied to the beam. The idea behind this was to keep the reactions vertical with no horizontal component. Also, to keep the distance between loads constant as the beam deflected under the applied loads. The knife-edges were made long enough to give good bearing on the cross-member channels. (See figure 13.)

LOAD APPLICATORS

The next problem attacked in the design of the testing apparatus was the method of applying the test loads to the beam models. This problem was overcome by designing hangers which would allow the load to be applied to the top of the beam while placing the weights on below the bench, which was much more accessible. The load applicator itself was a machined knife-edge which had a strap bent over the top of it, in the shape of an inverted U. The strap was bolted on to the knife-edge. Through the ends of the legs of the inverted U holes were drilled, through which a round bar was inserted. (See figure 13.) A hook hanger was hung over this round section. The lower end of this hanger was also hooked, and on this hook was placed the basket in which the actual loads were placed.

FIGURE 13



Assembled Loader
Full Scale
Drill for 1/4" Screws
both sides machined
Detail of Knife-edge Supports
Scale ~ 1/2" = 1"
Make 4
Drill and Tap for
1/4" machine screws
All sides machined
Detail for Loader and Hanger
Scale ~ 1/2" = 1"
Make 3
8"
6"
2"
2 1/4"
2 1/2"

TEST WEIGHTS

When it came to the actual testing, great difficulty was encountered in obtaining loads to use in loading the test sections. This problem was overcome through the aid of Dr. [illegible] of the Metallurgy Department. The weights finally used consisted of small cast lead ingots averaging about eight pounds in weight for the primary loads. Finer loading, used as the failure loads were approached, was made up of small lead pellets which weighed about four tenths of a pound each. Each of the ingots used was weighed previous to any testing, and its weight to the nearest tenth of a pound was recorded on the ingot so that the weights could be accurately accounted for as they were added to the loading.

DEFLECTION GAGES

Deflection gages, with holders, were obtained through the efforts of Prof. [illegible] of the Mechanics Department. The holders were clamped by means of small C-clamps to the flange of one of the beam cross-sections. The holders gave added versatility to use of the gages as they could be adjusted to any position desired to take necessary readings. The gages were used as an aid in observing when the formation of the plastic hinges occurred.

RIGID FRAME TESTS

Near the end of the experimental work it was decided to test some rigid frame possibilities in order to see whether or not the theory could be borne out with this type of

indeterminate structure. As has been noted in the theoretical discussion, the advantages of this plastic theory become more apparent as the degree of redundancy increases. A special apparatus had to be designed to enable us to adapt our methods of loading to rigid frames. Again the apparatus designed was designed for this special problem and its application limited in that respect. It was decided that the best span to use would be the one foot span which would simplify calculations as well as loading. Also the clamps were made to give as rigid a connection for the legs of the frame as could be had so that in every respect the rigidity and construction of the model would be as close as possible to the real thing in order that the data obtained from such a test would be worthwhile.

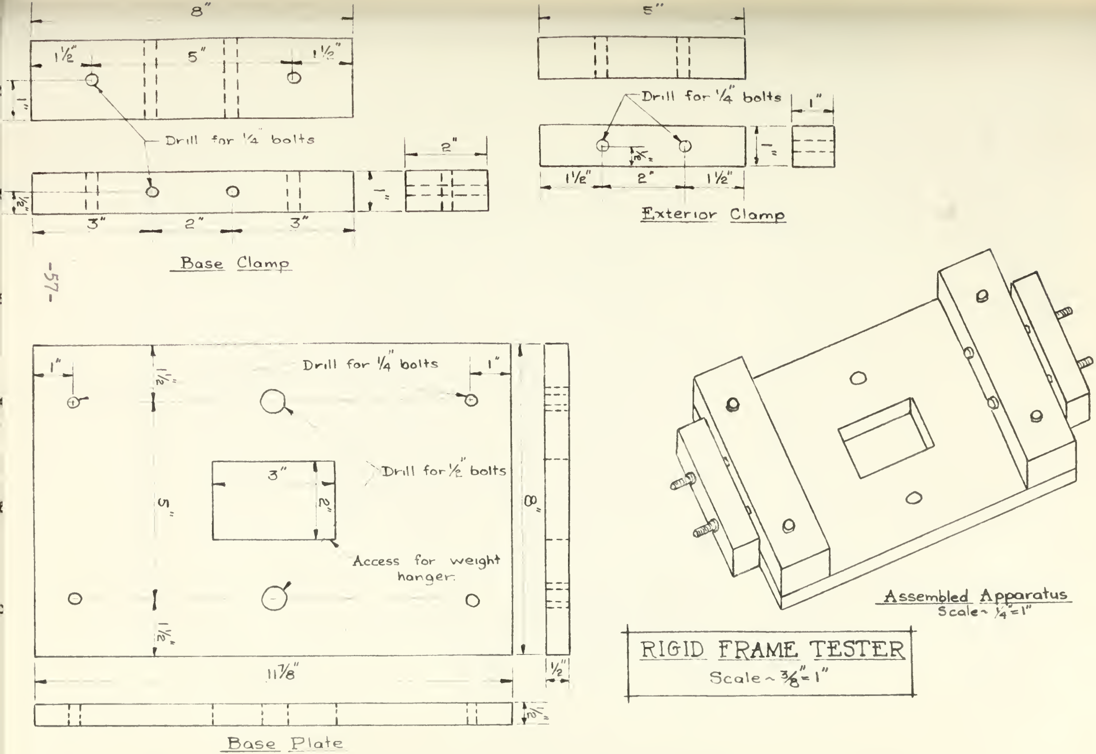
BASE AND CLAMPS

The base and clamp blocks were machined to give as smooth a bearing surface as possible. The base was drilled with holes for half inch bolts so that it could be bolted to the main testing bench in such a way as to allow the simulation of a side-load on the leg of the frame. The base was constructed so that it could be placed flat on the main test bench cross-members for testing of frames under the action of loads on the top or cross span of the frame. (See figure 14.)

LOADING AND GAGES

The loads were applied by the same loading used for

FIGURE 14



RIGID FRAME TESTER

Scale ~ ⅜"=1"

the beam tests. For testing of the top span, a hole was cut in the center of the beam through which the load hanger was hung. Deflection gages were used in the tests in order to show more closely when the plastic hinges formed anywhere in the frame. In the case of the top span loading only one gage was used, and that was placed over the load applicator. In the side-loading test, two gages were used, one on the load applicator and the other on the opposite leg near the center.

COMMENTS

The equipment designed proved to be entirely adequate for the scope of the work contemplated in this problem. However, it is not big enough nor versatile enough to test actual scale models of frames and other structures, the testing of which seems to be the next logical step in the development of this theory. For such work a static tester which would allow the application of several loads of different magnitudes, larger than anything used in this problem, and at the same time should be constructed. We understand that such a machine is in process in the Civil Engineering Department.

METHOD OF TESTING

As has already been set down in the scope of this problem, we limited our work to the investigation of the formation of plastic hinges in simple structures. We further limited our work to concentrated, static loads. Because of the time limitation it was felt that we could not do justice to any greater undertaking. The sections tested in experimental work were: simple beams, continuous beams, fixed-end beams, propped cantilever beams and rigid frames. Also tested was a continuous beam of circular cross-section to show the effect of concentration of the mass near the neutral axis of the beam on the resisting moment of the beam. This was done to carry through with the theory and show how the more complicated or varied the sections or structures become, the more advantage the plastic theory has over the commonly used elastic theory.

YIELD STRENGTH TESTS

In order to determine the yield strength of the material used in the experimental work, tensile test specimens were made up and tested with the machine available in the Mechanics Laboratory. Several tests were run and the results are shown in the appendix. On the basis of the yield strength thus determined, the theoretical failure loads of the beams were calculated.

SIMPLE BEAM TESTS

Before any tests were run on the continuous beams,

simple span beams were tested to determine experimentally the value of the plastic moment of the material being tested. This was done to verify the theoretical calculations of the plastic moment of the section being tested. The test was considered to be a more accurate indication of the plastic moment of the material, as it took into account the amount the cross-section of the material might have varied from the exact rectangular, which was the section assumed in the theoretical calculations.

LOAD POINTS

The concentrated loads used were placed at the midpoint of the spans as well as at other noted points along the span to see how well the actual loads would conform to the values predicted on the basis of the theoretical analysis of the same sections.

TEST SAMPLES

In the case of the continuous beams the span used was usually 12". This span was picked for simplification of mathematical analysis, ease of loading and also with the idea of keeping the L/b ratio reasonably low, to avoid incurring lateral buckling during the application of the loads. The span was about as short as could be used to keep the loads within the scope of the apparatus. The material used in all tests was mild steel, hot rolled. The sections used were all made of 1/8" by 1-1/8" strap. The one test on the circular cross-section used hot-rolled mild steel 3/8" in diameter.

BEAM SET-UP

No special preparation was necessary for the beam test samples. They were sawed from a long piece of stock. The beams were marked with a mark at the location of the supports and where the loads were to be applied. According to the theory being investigated, little attention had to be paid to such things as surface marks and scratches. As will be shown, this was borne out by the results of these tests. Care was taken to make sure that the knife-edge supports were placed at the proper location and were kept perpendicular to the sample at all times. Also, care was taken to keep the loading knife-edges at the proper location and the knife-edge perpendicular to the axis of the sample.

SETTING GAGES

The deflection gages were set on each of the load applicators. Care was taken to keep the plungers vertical and at the center of the loaders so that no error due to rotation or tilting of the loaders would be present in deflection readings. After the gages were set to zero, the hangers and load baskets were hung on the loaders. Readings of the gages were taken after each increment of load was applied.

APPLYING LOADS

The loads were added in equal increments to each end of the beam under test. The individual and accumulative loads were noted and recorded as each additional weight

were added. The primary loads were made up of the lead ingots which averaged about eight pounds each. As the total load was built up to near the point where the formation of a plastic hinge at some section in the beam was reached, the individual loads were changed to the smaller lead pellets. These weighed about four tenths of a pound each and they were added one at a time. The deflections were read and recorded only after three or four of the pellets had been added if no unusual deflection had taken place with the adding of the individual pellets. This was done so that an accurate curve of the deflections versus the applied loads could be plotted for each test, without having the points of the plot too close together. As the point in the loading near the formation of a plastic hinge at some section in the test structure was reached, the loading and the observation of the deflection gages were very carefully carried out. Extreme care had to be taken to make sure that the loading baskets were not allowed to swing back and forth in pendulum action, or to rock due to any vibration or other disturbances of the apparatus. In some test runs it was found that this rocking tended to speed up the formation of the plastic hinges or to give rather erratic readings, so it was necessary to be careful in this regard. The loading with the lead pellets was continued until the first plastic hinge formed. After the formation of the first hinge, the beam was allowed to deflect until it reached

the fully relieved state without further load. Then it was further loaded until the formation of the next hinge. This procedure was continued until the beam completely failed. This state being reached when the deflection under the load was no longer of definite duration, but continued on until the load was taken off or until it had been relieved in some external manner. This final load is termed in this paper "failure load", and is marked on the loading graphs included in the appendix.

CIRCULAR SECTION CONTINUOUS BEAM

As previously noted, a continuous beam of circular cross-section was tested. It was tested in the same manner as the other continuous beams, except the span was lengthened to 30", as the load to be handled with the shorter span would have been quite high. This longer span kept the load down, yet enabled full observation of the desired results. The beam was made up of 3/8" round bar, hot rolled or drawn. A tensile test was also run on this material to determine its yield strength so that the experimental results could again be compared with the theoretical. The same load applicators were used with this circular section and care had to be taken to see that they were properly balanced on the beam. This did not turn out to be as difficult as had been anticipated. It was thought that the extremely small bearing area of the knife-edge would allow the beam to roll and thus cause the loads to move off center and cause us to lose control

of the loading. This did not occur. As soon as any load was applied the beam became very unstable, and increased in instability as the load, and consequently the deflection increased. Initially, the beam was restrained from rotation by wrapping a small wire around the support knife-edges on either side of the beam. This was found to be very satisfactory as it did not interfere with the natural performance of the beam under the loading, but did prevent it from rotating.

FIXED-END BEAMS

The fixed-end beam set-up was simulated by first using the two clamps of the frame testing apparatus and in turn clamping them to the cross-members of the testing board at a distance apart which would allow for the testing of the desired length of beam. The ends of the beam to be tested were clamped into the clamps very tightly so as to produce a rigid and fully fixed condition of connection. The load was then applied as in the case of the continuous beams. This set-up was found to be unsatisfactory because as the beam deflected, the rigid connections would not allow the ends of the beams to pull toward each other, thus introducing an axial tensile stress into the member as well as the bending stress which is what was under investigation in this problem.

An attempt to overcome this problem was made by utilizing two of the knife-edges at each end of the beam being tested. These supports were placed about two inches

apart and the beam passed over them. On top of the beam the main clamp was clamped down. This simulated a fixed end beam, but allowed the ends to move toward each other, keeping constant distance between the two ends of the beam and in this way the axial stress was eliminated satisfactorily.

PROPPED CANTILEVER BEAMS

The fixed end of the propped cantilever was set up in the same way as in the above experiments with the fixed-end beams. At prescribed distances from the fixed end of the beam the knife-edge support was placed for the simply supported end of the beam. The loading was carried out in the same way as in the previous beam tests. In these experiments and in those with the fixed-end beams only one deflection gage was used, and it was placed over the load application in a manner similar to that noted in the previous explanation of the tests on the continuous beams and frames.

OBSERVATIONS MADE

After the beam had been loaded to failure, the loads were removed and the final attitude of the bent beam noted and recorded. More reference will be made to these tested beams in a later section of this paper. Photographs of some of the tested beams will be seen in the appendix. During all of the tests, the behavior of the beams as they were loaded was observed. Especially so were all observations during the time when the plastic hinges were forming, or

were about to form.

FRAMES

The frames were made up of the same material used in the beam tests. They were made by bending a strip of the strap into the U shape of the frame by heating the strap at the desired points with an acetylene torch and bending while still hot. The top span was made 12" long. The legs of the frame were clamped into the special tester in such a way that the two legs were each 12" long. Care was taken to make sure that the clamps were as tight as possible so that the frames would behave as rigid frames.

LOADING OF FRAMES

The loads of the frames were built up in the same way as noted for the tests on the beam sections. In these frame tests, the loading was a bit different because of the fact that there were several hinges formed, and after the formation of the first hinge additional load could be applied until the whole frame was turned into a mechanism as defined in the theoretical discussion. As each plastic hinge formation was approached in the loading, the loads added were changed from the iron ingots to the lead pellets as noted before in the beams tests. The difference in these tests being that after the plastic hinge was formed, quite a bit of additional load could be applied and this was accomplished by resuming loading with the ingots until nearing the formation of another

plastic hinge when the load points were again referred to.

DETERMINING PLASTIC HINGE FORMATION

The frames were tested with only one load at a time. Some tests were made with top-span loading and others with side loading. The loads were applied in the same manner as set forth in the discussion of the tests on the beam sections. In the frame tests with side loading (figure 19 shows one in progress) two deflection gages were used. One was placed over the load application. The second was placed in a like position on the unloaded leg. These two gages were used in hopes of determining exactly where and when the plastic hinges formed in the frame under the test loads. It was hoped that the sequence of the formation of the plastic hinges, if there were to be any particular sequence, could be observed in this way. Another evidence of the formation of plastic hinges watched for and not mentioned in the previous discussion was the cracking of the scale at the section where the plastic hinge formed. This was especially watched for in these frame tests, because, as mentioned above, we were desirous of determining the sequence of the formation of the plastic hinges. Simply observing the deflection gages would not give this information.

COMMENTS ON TEST DATA

It might be noted here that in the theoretical calculations the point usually determined was the ultimate

failure point, that is to say, the point where all the plastic hinges that are going to form have formed and the structure is rendered to the state of a "mechanism". In most cases these hinges formed at separate points in the loading of the beam. In our first tests with each structure we ran one test for a dummy run to determine when these intermediate hinges would form and then ran our final tests utilizing the information thus determined. In this way it was determined when to change from the coarse loading with the lead ingots to the finer loading with the lead pellets so that a more accurate determination of the point of formation of the plastic hinges could be had. Later on in our testwork it was found that we could calculate from static formulas, based on the plastic theory, when these intermediate hinges would form. Then it was not necessary to run two runs of each test set-up but we could run our one run through on the basis of our mathematical calculations and then be aware of when to expect the formation of these intermediate hinges, and introduce the finer loading at the proper time.

RESULTS

TENSILE TESTS

The mean of the tensile tests, which incidentally were all very close, indicated that the material used in our tests with the beams and frames had a yield strength of 38,350 psi. This value of yield strength gave a theoretical value of plastic moment equal to 168.53 in-lb.

SIMPLE BEAM TESTS

The mean of the loads on the simple beam tests, which were run to give the experimental value of the plastic moment of the material being tested, was 99.42 lbs. This load gave an experimental value of the plastic moment equal to 169.26 in-lb. This compared very favorably with the theoretical value of 168.53 in-lb. The difference amounted to 0.43% of the theoretical value. We suggest that this small error might very well be due to the small variation from the rectangular which the actual cross-section of the material had. Our theoretical values were based on an assumed rectangular cross-section.

It was noted in the simple beam tests that the beams behaved in the manner which proved to be usual for the rest of our tests. We were able to observe the formation of the plastic hinge under the load, as we had calculated it would form. As the load was applied to the beam it deflected in the normal manner. As the beam deflected it displayed a deflected structure in

the form of a very smooth curve. This curve was manifest until the instant the plastic hinge formed. At this time the curvature completely disappeared and the sections on either side of the point where the plastic hinge formed, assumed perfectly straight line attitude. Attention is called to the accompanying figure which shows the load versus deflection curve, which is a composite plot of the tests that were run on the single beams. Note especially that the curve is a straight line all the way to failure. This development was rather unexpected, as we had thought that the curve would show a definite trend away from the straight line after the yield stress and moment as determined by the triangular stress distribution pattern, and as noted on the figure, was passed. This was not the case, as the graph shows. When the plastic hinge formed, there was noticeable chipping off of the scale on the member at the point where the plastic hinge formed. Another unexpected result was the very narrow band within the effective distance of the plastic hinge. This may be noted on the figure showing the beams which have been carried through the test to failure. At the very instant the plastic hinge formed in the material there was a slight metallic "ping" after which the deflection suddenly increased at a very rapid rate. In the case of the single span the beam kept on failing until the load was relieved. Thus, the idea of a mechanism was borne out, though later

experiments with the load increment structure showed this much more graphically. As the loads were added up until failure (as the graph shows) the deflection under each increment was a very definite amount. Under the failure load deflection continued on indefinitely without any additional load being applied to the member.

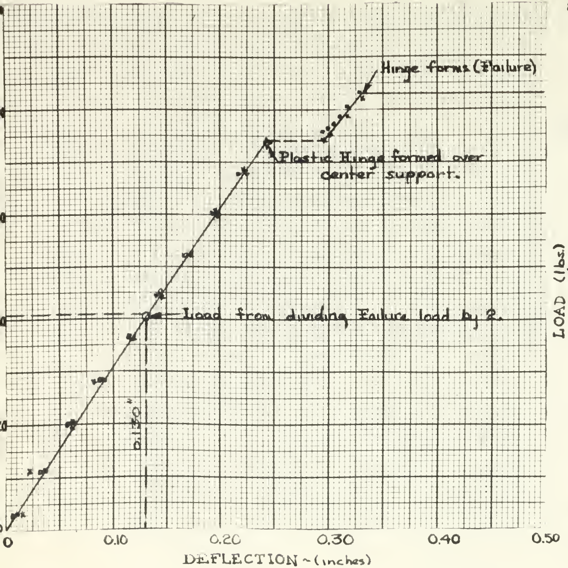
TWO-SPAN CONTINUOUS BEAM

The accompanying graph shows the results of the load tests performed on the two-span beams. Note especially the comparison graph (figure 17) which indicates how the tests conducted at different points along the spans of the beams agreed with the predicted results of such loadings. Notice also the plot of load versus deflection (figure 15). This indicates that the deflection of the span of the two-span beam under the limit load, divided by a suitable safety factor of two, is less than the deflection of the single span under the normal elastic load for which it would be designed. Said deflection of the span of the two-span beam is 0.130", while the deflection of the single span beam under similar loading (load applied at the mid-point) is shown to be 0.140". This bears out the argument presented in the theoretical discussion.

LOAD APPLIED AT THE MID-POINT

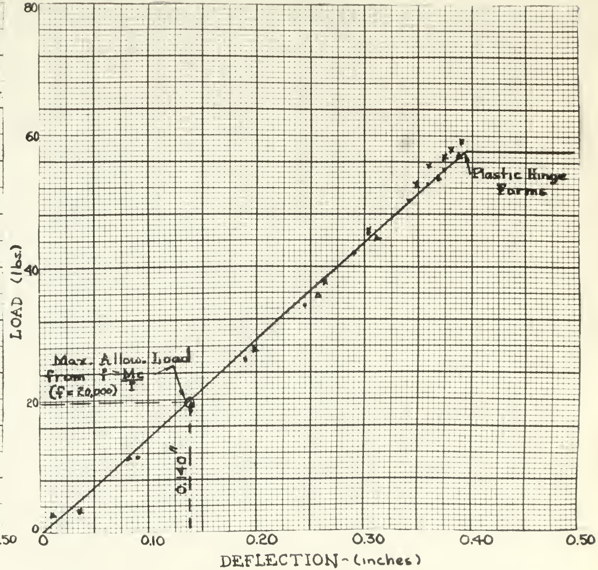
Figure 15 shows the results of this test run on three different samples. The results were very close in all cases. Under this condition of loading, the first plastic hinge formed over the center support. This led

FIGURE 15



Loading Results from Tests on 2-Span Continuous beams loaded at midpoints. Deflection $P_F/2 = 0.130''$.

FIGURE 16



Loading Results from Tests on Simple Beams, loaded at midpoints. $P_F = 56.42$ lbs. $M_P = 169.26$ in-lb. Deflection by Elastic design $(f = Mc/I) = 0.140''$.

been predicted from the use of the regular beam formulas. This first plastic hinge formed with a load of 74.0 lbs. At this point a very definite rapid deflection took place. The beam became stable under this load, and more load was added with the beam continuing to exhibit strength. The plastic hinges then formed under the loads and the beam became a mechanism, and, without the addition of further load, continued deflecting until failure. This ultimate failure took place under a load of 83.6 lbs. The failure load predicted for this condition of loading was 84.2 lbs. The difference was 0.71% of the predicted load. The predicted load for the formation of the first plastic hinge which formed over the support was 6/32 Pl which equaled 74.8 lb. The difference in this case amounting to 1.06%.

LOAD APPLIED AT THE .415L POINT

After the first few tests were run and the behavior of the beam became a little more predictable with equations, it was decided to try to determine the point of loading on the spans of the continuous beam that would give a condition of simultaneous failure. That is where the plastic hinges would form under the loads and over the interior support at the same instant. This loading was predicted by solving the moment equations simultaneously. Our mathematical determinations showed that this type failure could occur if the spans were loaded at a distance of 0.415 x l from the simply supported ends of the continu-

ous spans. Incidentally, this theoretical calculation was borne out almost exactly by the graph plotted, showing theoretical values of X versus the constant K. (figure 6). As may be noted on that graph, this point is the low point on the curve, and within the accuracy of the curve can be read to give 0.42L. This loading condition reacted in almost exactly the way that had been predicted for it. The load at failure was 61.25 lbs, as compared with the theoretical load of 61.50 lbs. This represented a difference from the theoretically calculated value of 0.92%. Also of note is the fact that all the hinges did form simultaneously just as had been predicted. The beam was perfectly elastic in behavior up until the failure load was put on the beam. At that time, the plastic hinges formed simultaneously under the loads and over the center support and the structure was transformed into a mechanism and continued to deflect until the load was relieved. Also, as might be shown from the theoretical graph and as borne out by the experimental graph, this loading point was found to be the weakest for the continuous beam. In other words, the failure load that was put on the beam to induce its failure with loading at the 0.415L point was the least load that it took to produce failure in the continuous two-span beam.

LOAD APPLIED AT THE 0.6 L POINT

Our theoretical predictions on this condition indicated that the beam would fail over the mid-support

first and then under the loads. The failure load predicted from calculation in this case was 94.8 lbs. The beam tested actually yielded over the mid-support first, under a load of about 75.7 lbs. Then the beam strengthened and behaved elastically again until the plastic hinges formed under the loads and transformed it into a mechanism. This failure load was 94.8 lbs., exactly as predicted in one test, and 95.0 lbs., a difference of 0.21% from the theoretical in the other test that was run under the same conditions of loading. It might be well to emphasize here again that the formation of these plastic hinges was very easy to observe with the experimental set-up that was used to determine them. The first notice that was given was when the deflection dial would start moving at an unusual rate compared to what it had been doing under the loads up to this point. Then at the same instant the noticeable "ping" of the metal could be heard. Simultaneously, the chipping off of the scale on the tension side of the beam at that point could be observed. Thus, there were three very distinct manifestations of the formation of the plastic hinges in the material which would be observed so that these particular points were easily determined.

LOAD APPLIED AT THE 0.7L POINT

As may be noted on the curve (figure 8), this loading condition indicated again that the first plastic hinge would form over the mid-support and then the hinges would

form under the loads. The failure load predicted for this loading condition was 112 lbs. Under actual loading, the first plastic hinge formed over the mid-support under a load of 93.7 lbs. The plastic hinges formed under the loads, with resulting failure of the structure, under a load of 110 lbs. This gave a difference of 1.79% from the theoretical.

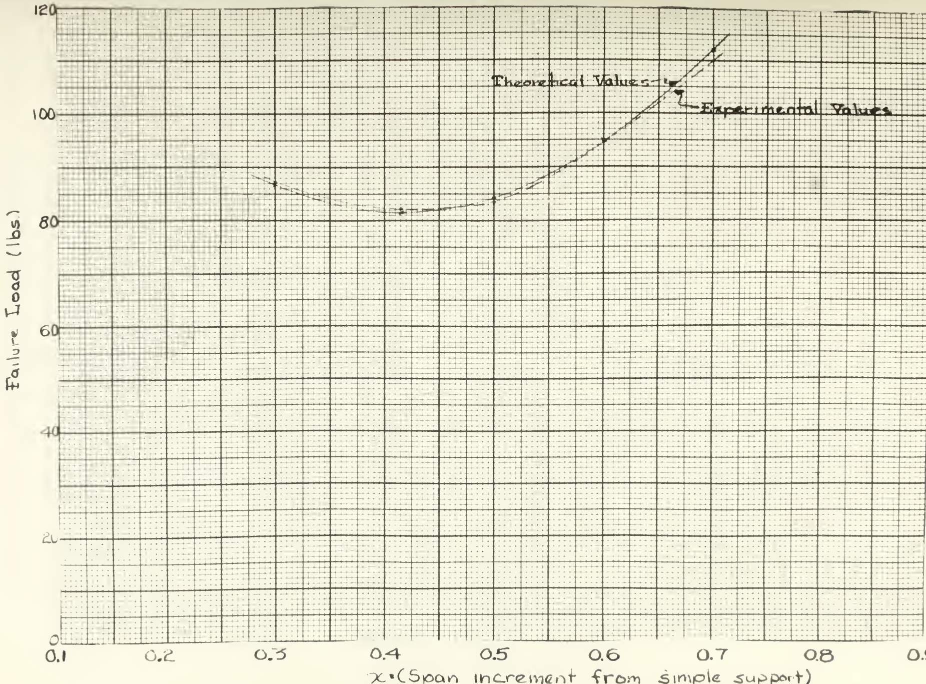
LOAD APPLIED AT THE 0.3L POINT

From the curve of theoretical values (Figure 6), a failure load of 87.0 lbs. was predicted for this condition of loading. Also, from the curve, and the behavior of the previous beams tested, it was anticipated that the plastic hinges in this case would form under the loads first, with ultimate failure resulting when the plastic hinge formed over the mid-support. The actual loading bore out the anticipated behavior. The plastic hinges formed under the loads first, at a load of [illegible].1 lbs. After the rapid deflection at this point in the loading the beam stiffened and did not fail until the plastic hinge formed over the mid-support, which occurred under a load of 87.35 lbs. This represented a difference from the theoretical of 0.40%.

3/8" ∅ BAR - LOAD APPLIED AT THE 0.415L POINT

From the theoretical curve (Figure 6), it was predicted that the plastic hinges would form over the mid-support and under the loads at the same time. As it may be noted, the 0.415L point is the low point on the curve.

FIGURE 17



Comparison of Theoretical and Experimental values of Failure Loads for 2-span beam loaded at various span increments.

Also, from the theoretical calculations, it was predicted that the failure load would be 143.81 lbs. It was further predicted, on the basis of the theory previously developed in this paper, that the fact that this beam was of circular cross-section would in no way detract from the fact that the behavior under loading would be essentially the same as that of the beams which were of rectangular cross-section. Under the actual loading test of these expectations was borne out with remarkable exactness. The actual failure load of the beam was 143.25 lbs., which represented a difference from the theoretical of only 0.39%. The behavior of the beam under loading was very much like that of the rectangular beams, in fact there was no difference which could be observed. The plastic hinges did actually develop at the same instant, and in the same manner as had been noted in the previous rectangular beam tests. The scale chipped off on the tension side at the hinge, and after the hinges formed there was that definite deformation which could be observed in the deflected beam shown in figure [illegible]. The parts of the beam between the hinges remained elastic, as was borne out by the fact that they were perfectly straight.

An interesting result which might well be included here is the fact that relatively, the circular section has even more advantage, as determined by the plastic theory, over the elastic theory for the same section than does the rectangular section. The circular section gives a

ratio of $\sum(ay)/Sx$ equal to 1.70, while the same ratio for the rectangular section is 1.50. This indicates that the fact that the mass of the section were concentrated about the neutral axis of the member does increase the effectiveness of the plastic method, as the circle is the one extreme of that condition. Conversely, as is pointed out in the discussion of theory, the I-beam, which could be used as the other extreme, shows a ratio in a similar comparison of only 1.05 to 1.15.

FIXED-END BEAM TESTS

These tests proved to be very unsatisfactory and inconclusive, mainly because of the difficulty of simulating the fixed-end condition. With the first set-up it was found that the actual loads exceeded greatly the loads which had been predicted for such a condition and set-up. With the second set-up, which more closely approximated an actual fixed-end beam in our minds, the results were much better, giving a failure load of 107.85 lbs., versus the predicted load of 111.2 lbs., which represented a difference from the theoretical of 4.96%. While this would seem to be a satisfactory result, and to our minds is enough to bear out the theory for this structure, it is felt that these particular tests could be set-up in a more satisfactory manner to yield more conclusive results.

THIN FRAME - LOAD RATIO OF 0.5 OR [illegible]

According to the theory, hinges would develop under this condition of loading at A, B, D & E on the frame

shown in figure 10. The frame would not fail until these four hinges formed, and they would all be formed when the load reached a value of 8[illegible] lbs. Under the loading, the first hinge formed at B under a load of 53.75 lbs. The second hinge formed at A at a load of 80.0 lbs., and a hinge formed at D almost at the same time. The structure still retained strength after these three hinges had formed, and supported added loads up until the fourth hinge formed at C under a load of 83.15 lbs. At that time, the structure was transformed into a mechanism and continued to deform until the load was released. This load represented a difference of [illegible].9% from the theoretical.

The hinge which formed at the corner of the frame B, did not form right at the corner in any of the experiments that were run under this condition of loading. It was thought, and seemed obvious from the appearance of the hinges on the deflected frames that this might have been caused by the fact that the character of the material in the frame was altered right at the corner by the action of the heat and bending which was done to make the frame. This difference might be rectified by heat treating the frames after they are bent and formed. The present authors were prevented from following this through by a lack of time in completing the project.

FIXED FRAME - LOAD APPLIED AT 0.5L OR MID-SPAN

According to the theory, hinges would develop under the load and at both ends of the span. Also, the bar

legs would be subjected to the plastic moment, or very nearly its equal, throughout their entire length. This is what the theory showed, and we were very skeptical of it, as it is a rather radical departure from what the elastic theory shows to be the case for the same structure.

The behavior of this structure under the loading turned out to be very interesting. As the first loads were added, the deflected structure appeared to be very similar to what would be expected from an elastic analysis of such a frame. During the initial loading all the members of the frame deflected and were curved in smooth curves. The two legs each exhibited a point of contraflexure that was noted to move down the leg toward the fixed-end as the load increased. Finally the point of contraflexure disappeared and the whole member was just one smooth curve. When the first hinge formed, it formed under the load. This happened under a load of 86.75 lbs. The next two hinges formed simultaneously and they formed under a load of 1[illegible]3.75 lbs. It was at this time that the legs of the structure showed no point of contraflexure in them, and exhibited the form of a smooth curve. The frame was in a practical failure condition at this time. A little more load was added, but it was impossible to keep track of the deflections past this point with the set-up that we had. It was noticed, however, that this was very near the complete failure condition that had been predicted

for this structure. This was indicated, when another hinge was formed down on the leg of the frame with a very little addition to the load. This seemed to indicate very well that the full length of the legs were being subjected to about the plastic moment at this time, as this additional plastic hinge formed at no particular point but just between the corner and the fixed-end of the leg. Even so, the failure load was observed to be only 1.9% off from the load which had been predicted for the structure. This error might have been cut down further had we been able to follow through the loading as desired to full failure of the structure. It is felt that the loading was carried out far enough and the observed results conclusive enough to bear out the theory very well on this particular structure.

Here again the temperature and bending effects were noticed in the material. The hinges did not form exactly at the corners, but rather some distance down the leg from the corners and right at the edge of the heat affected zone of the metal.

COMMENTS ON FRAME TESTS

In both these structures the hinges showed up the same way they did in the simpler beam tests. There was a noticeable movement of the deflection gauges, accompanied by the "pinging" sound and the flaking of the scale from the tension side of the member at the point where the plastic hinge formed. After being removed from the

testing machine, the deformed structure exhibited the same characteristics which were noted in the test with the beam sections. The sections of the material between the hinges seemed to be unaffected and remained perfectly straight. Also to be noted in all of these tests, and as may be seen from the accompanying photographs of the deformed structures, (figure 30), the zone which was included in the area of the plastic hinge was very small in every case. Also, in every case the structure behaved completely elastically up until the time the first plastic hinge formed, and in the case where more than one hinge was formed, continued to act elastically up until the final hinges were formed, with the only departures arising at the points where the hinges were formed in the structures. This bears out the point brought out in the theoretical discussion, namely, that there exists in the material an elastic core which enables the overall structure to act elastically until that core is eliminated as the stress distribution approaches the rectangular pattern.

CONCLUSIONS

The authors of this thesis, on the basis of the theoretical development and experimental results set down herein, feel that the following conclusions may be drawn from the work done:

1. Such phenomena as described in the theory (the formation of plastic hinges) does occur in actual practice, and the moment present at the point of formation of the plastic hinge, termed the plastic moment in this paper, is a definite characteristic of ductile materials, dependent on the yield strength of the material and the shape and size of the section used.

2. Simple redundant structures may be accurately analyzed for failure loads by means of assuming plastic hinges at certain points in the structure. This allows the structure to be broken down to free bodies which can then be solved by the formulas of statics.

3. Such an analysis as outlined above gives a design based on a definite criterion, namely failure, and by using a suitable safety factor, one can design with a definitely known margin of safety against failure.

4. Deflections caused by the loads determined by the above method, divided by a reasonable safety factor, are comparable to elastic deflections, and in the case of redundant beams, are less than the deflections caused by loads allowed on simple spans by the elastic theory. At the same time, this means safe load as determined by this plastic theory is enough larger than that allowed by

the elastic theory to indicate a definite economy of design that might be had by using the plastic theory.

5. The structure acts elastically until the formation of the first plastic hinge, and even then only a very small section of the structure right at the point where the plastic hinge is formed is inelastic. The rest of the structure or member remains elastic.

RECOMMENDATIONS FOR FUTURE INVESTIGATIONS

The authors of this thesis realize that the work herein described only scratches the surface of the plastic theory. However, we feel that the results which we have achieved warrant carrying on the work with an enlarged scope and objective. Nothing was found in any of the investigations which were attempted and carried through that did anything but encourage the idea that there is definitely something in this theory. The following recommendations for future investigations are made in hope that someone will carry on from the modest beginning that has been made.

1. The scope of this paper should be enlarged on by:

   a. Investigating other sections, or models of them, such as I-beams and WF sections.

   b. Investigating the effect of multiple loads.

   c. Investigating the effect of uniform loading.

   d. Investigating the effect of dynamic loading.

   e. Investigating actual scale models of redundant structures such as rigid frames.

2. Investigate concrete sections to see whether or not the theory developed in this paper can be applied to them, especially with regard to time lag effect of stress relief which is believed to be inherent in concrete structures.

3. Investigate other structures which are more complicated and more redundant than those analyzed and tested in this thesis.

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Brown University

Providence, Rhode Island

October 1949

APPENDIX

EXPERIMENTAL DATA

FIGURE 18

FIGURE 19

FIGURE 20

EXPERIMENTAL DATA

RUN #1 -- simple beam, load at mid-point, 12" span.

Weight	Accumulated Weight	Dial Reading	Deflection
0.00	0.00	900	-----
3.45	3.45	864	0.036
34.20	37.65	637	0.263
8.00	45.65	597	0.303
7.00	52.65	551	0.349
2.05	54.70	540	0.360
2.25	56.95	525	0.375
1.25	58.20	519	0.381
0.95*	59.15*	512*	0.388*

RUN #2 -- simple beam, load at mid-point, 12" span.

Weight	Accumulated Weight	Dial Reading	Deflection
0.00	0.00	900	-----
3.45	3.45	864	0.036
8.00	11.45	811	0.089
7.00	18.45	761	0.139
7.50	25.95	712	0.188
8.00	33.95	655	0.245
8.00	41.95	609	0.291
8.00	49.95	555	0.345
4.60	54.55	526	0.374
1.50*	56.05*	---	-----

RUN #3 -- simple beam, load at mid-point, 12" span.

Weight	Accumulated Weight	Dial Reading	Deflection
0.00	0.00	800	-----
3.45	3.45	778	0.022
8.00	11.45	719	0.081
8.00	19.45	660	0.140
8.00	27.45	601	0.199
8.00	35.45	544	0.256
8.00	43.45	476	0.324
7.90	51.35	428	0.372
2.30	53.65	411	0.389
2.00*	55.65*	394*	0.406*

(*) indicates failure load.

EXPERIMENTAL DATA

RUN #4 -- simple beam, load at mid-point, 12" span.

Weight	Accumulated Weight	Dial Reading	Deflection
0.00	0.00	800	-----
3.45	3.45	778	0.022
8.40	11.85	716	0.084
8.30	20.15	655	0.145
8.30	28.45	594	0.206
8.30	36.75	534	0.266
8.30	45.05	472	0.328
8.15	53.20	412	0.388
3.95	57.15	381	0.419
0.40*	57.55*	377*	0.423*

(*) indicates failure load

EXPERIMENTAL DATA

RUN #5 -- continuous beam, two spans, loads at mid-points, 12" span.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	800	-----	800	-----
3.45	3.45	790	0.010	790	0.010
8.30	11.75	777	0.023	777	0.023
8.30	20.05	735	0.065	735	0.065
8.20	28.25	708	0.092	708	0.092
8.20	36.45	682	0.118	681	0.119
8.00	44.45	656	0.144	655	0.145
8.00	52.45	630	0.170	629	0.171
7.80	60.25	603	0.197	602	0.198
7.80	68.05	577	0.223	578	0.222
7.70**	75.75**	[illegible]**	0.30[illegible]**	501**	0.299**
5.05*	80.80*	470*	0.330*	470*	0.330*

RUN #6 -- continuous beam, two spans, loads at mid-points, 12" span.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	600	-----	600	-----
3.45	3.45	590	0.010	590	0.010
8.30	11.75	564	0.036	563	0.037
8.30	20.05	538	0.062	536	0.064
8.20	28.25	512	0.088	509	0.091
8.20	36.45	486	0.114	486	0.114
8.00	44.45	460	0.140	456	0.144
8.00	52.45	435	0.165	432	0.168
7.80	60.25	410	0.190	406	0.194
7.80	68.05	385	0.215	381	0.219
7.20**	75.25**	308**	0.292**	304**	0.296**
1.20	76.45	302	0.298	298	0.302
1.20	77.65	297	0.303	293	0.307
1.20	78.85	291	0.309	286	0.314
1.20*	80.05*	283*	0.317*	278*	0.322*

(**) hinge forms over support.
(*) indicates failure load.

EXPERIMENTAL DATA

RUN #7 -- continuous beam, two spans, loads at mid-points, 12" span.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	800	-----	800	-----
3.45	3.45	792	0.008	792	0.008
8.30	11.70	767	0.033	767	0.033
8.30	20.00	743	0.057	743	0.057
8.20	28.20	720	0.080	719	0.081
8.20	36.40	695	0.105	695	0.105
8.00	44.40	676	0.124	672	0.128
8.00	52.40	652	0.148	648	0.152
7.80	60.20	628	0.172	626	0.174
7.80	68.00	606	0.194	603	0.197
7.20	75.20	584	0.216	582	0.218
2.40**	77.60**	534**	0.266**	529**	0.271**
1.60	79.20	525	0.275	523	0.277
3.20	82.40	511	0.289	507	0.283
1.20*	83.60*	500*	0.300*	495*	0.305*

RUN #8 -- continuous beam, two spans, 12" span, loads at .415L.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	700	-----	700	-----
3.40	3.40	690	0.010	690	0.010
8.30	11.70	662	0.038	661	0.039
8.30	20.00	633	0.067	633	0.067
8.20	28.20	605	0.095	604	0.096
8.20	36.40	578	0.122	577	0.123
8.00	44.40	550	0.150	549	0.151
8.00	52.40	523	0.177	523	0.177
7.80	60.20	497	0.203	496	0.204
7.80	68.00	470	0.230	469	0.231
7.70	75.70	444	0.256	442	0.258
4.65	80.35	426	0.274	426	0.274
1.10	81.45	422	0.278	422	0.278
0.80*	82.25*	416*	0.284*	417*	0.283*

(**) hinge forms over support.
(*) indicates failure load.

EXPERIMENTAL DATA

RUN #9 -- continuous beam, two spans, 12" span, loads at .6L.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	700	———	701	———
3.40	3.40	691	0.009	692	0.009
8.30	11.70	673	0.027	673	0.027
8.30	20.00	653	0.047	654	0.046
8.20	28.20	635	0.065	636	0.064
8.20	36.40	617	0.083	618	0.082
8.00	44.40	599	0.101	600	0.100
8.00	52.40	582	0.118	583	0.117
7.80	60.20	564	0.136	565	0.135
7.80	68.00	546	0.154	548	0.152
7.70**	75.70**	529**	0.171**	531**	0.169**
7.50	83.20	450	0.250	452	0.249
7.20	90.40	415	0.285	416	0.284
3.60	94.00	401	0.299	401	0.299
0.20*	94.20*	394*	0.306*	394*	0.306*

RUN #10 -- continuous beam, two spans, 12" span, loads at .6L.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	700	———	700	———
3.40	3.40	692	0.008	692	0.008
8.30	11.70	673	0.027	673	0.027
8.30	20.00	655	0.045	654	0.046
8.20	28.20	637	0.063	635	0.065
8.20	36.40	618	0.082	617	0.083
8.00	44.40	601	0.099	599	0.101
8.00	52.40	584	0.116	581	0.119
7.80	60.20	566	0.134	563	0.137
7.80	68.00	549	0.151	545	0.155
7.70**	75.70**	531**	0.169**	527**	0.173**
———	———	485	0.215	480	0.220
7.50	83.20	450	0.250	443	0.257
7.20	90.40	411	0.289	404	0.296
4.60*	95.00*	380*	0.320*	370*	0.330*

(**) hinge forms over support.
(*) indicates failure load.

EXPERIMENTAL DATA

RUN #11 -- continuous beam, two spans, 12" span, loads at .7L.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	800	-------	800	-------
3.40	3.40	793	0.007	792	0.007
8.30	11.70	781	0.019	781	0.019
8.30	20.00	769	0.031	769	0.031
8.20	28.20	757	0.043	757	0.043
8.20	36.40	745	0.055	745	0.055
8.00	44.40	734	0.066	733	0.067
8.00	52.40	723	0.077	721	0.079
7.80	60.20	712	0.088	710	0.090
7.80	68.00	701	0.099	699	0.101
7.70	75.70	689	0.111	688	0.112
7.50**	83.20**	619**	0.181**	618**	0.182**
7.30	90.50	588	0.212	588	0.212
7.20	97.70	551	0.249	550	0.250
7.70	105.40	515	0.285	515	0.285
3.20	108.60	502	0.298	502	0.298
1.20*	109.80*	----	-------	----	-------

RUN #12 -- continuous beam, two spans, 12" span, loads at .3L.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	800	-------	800	-------
3.40	3.40	792	0.008	792	0.008
8.30	11.70	768	0.032	767	0.033
8.30	20.00	741	0.059	741	0.059
8.20	28.20	715	0.085	716	0.084
8.20	36.40	690	0.110	693	0.107
8.00	44.40	666	0.134	669	0.131
8.00	52.40	641	0.159	645	0.155
7.80	60.20	618	0.182	621	0.179
7.80	68.00	594	0.206	599	0.201
7.70	75.70	571	0.229	576	0.224
6.40***	82.10***	553***	0.247***	557***	0.243***
------	------	296	0.504	297	0.503
2.40*	84.50*	284*	0.516*	288*	0.512*

(**) hinge forms over support.
(*) indicates failure load.
(***) hinge forms under load.

EXPERIMENTAL DATA

RUN #13 -- continuous beam, two spans, 15" span, loads at .3L

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	700	------	700	------
3.40	3.40	692	0.008	692	0.008
8.30	11.70	668	0.032	670	0.030
8.30	20.00	645	0.055	646	0.054
8.20	28.20	622	0.078	625	0.075
8.20	36.40	600	0.100	603	0.097
8.00	44.40	579	0.121	582	0.118
8.00	52.40	557	0.143	560	0.140
7.80	60.20	536	0.164	539	0.161
7.80	68.00	515	0.185	518	0.182
7.70	75.70	496	0.204	498	0.202
7.00***	82.70***	474***	0.226***	478***	0.222***
4.65*	87.35*	461*	0.239*	464*	0.236*

RUN #14 -- continuous beam, two span, 15" span, loads at .415L, beam of circular cross-section 3/8" ∅ bar.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	700	------	700	------
3.45	3.45	691	0.009	691	0.009
8.30	11.75	669	0.031	668	0.032
8.30	20.05	645	0.055	644	0.056
8.20	28.25	622	0.078	621	0.079
8.20	36.45	600	0.100	599	0.101
8.00	44.45	579	0.121	576	0.124
8.00	52.45	556	0.144	554	0.146
8.40	60.85	533	0.167	531	0.169
7.80	68.65	511	0.189	509	0.191
7.80	76.45	489	0.211	487	0.213
7.70	84.15	468	0.232	466	0.234
7.70	91.85	444	0.256	444	0.256
7.50	99.35	425	0.275	423	0.277
7.30	106.65	403	0.297	401	0.299
7.20	113.85	380	0.320	378	0.322
7.20	121.05	356	0.344	353	0.347
[illegible]	[illegible]	312	0.388	312	0.388
7.10*	143.25*	---	------	---	------

(***) hinge forms under load.
(*) indicates failure load.

EXPERIMENTAL DATA

RUN #15 -- rigid frame, side loaded, load at mid-point, 12" span.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	1000	-----	1000	-----
3.25	3.25	985	0.015	991	0.009
8.30	11.55	929	0.071	963	0.037
8.30	19.85	872	0.128	934	0.066
8.30	28.15	816	0.184	902	0.098
8.30	36.45	758	0.242	872	0.128
8.20	44.65	702	0.298	847	0.153
8.70*	53.35*	543*	0.457*	772*	0.228*
8.00	61.35	435	0.565	721	0.279
8.00	69.35	308	0.692	661	0.339
8.00	76.85	185	0.815	603	0.397
4.60**	81.45**	----	-----	356**	0.644**
1.20***	82.65***	----	-----	----	-----
0.40	83.05	----	-----	----	-----
1.20****	84.25****	----	-----	----	-----

RUN #16 -- rigid frame, side loaded, load at mid-point, 12" span.

Weight	Accumulated Weight	Dial Reading	Defl.	Dial Reading	Defl.
0.00	0.00	900	-----	900	-----
3.75	3.75	873	0.027	890	0.010
8.30	11.65	828	0.072	860	0.040
8.30	19.95	774	0.126	828	0.072
8.30	28.25	719	0.181	796	0.104
8.30	36.55	663	0.237	764	0.136
8.30	44.85	607	0.293	731	0.169
4.60	49.45	586	0.314	713	0.187
3.60*	53.05*	550*	0.350*	----	-----
-----	53.05*	430*	0.470*	643*	0.257*
-----	-----	set 730*	-----	----	-----
8.00	61.05	650	0.580	587	0.313
8.00	69.05	496	0.734	523	0.377
8.00	77.05	380	0.850	457	0.443
2.00**	79.05**	----	-----	388**	0.512**
1.20***	80.25***	----	-----	----	-----
3.60****	83.85****	----	-----	----	-----

(*) denotes the formation of the first plastic hinge at B.
(**) denotes the formation of the second plastic hinge at A.
(***) denotes the formation of the third plastic hinge at D.
(****) denotes failure of the structure with fourth plastic hinge.

EXPERIMENTAL DATA

RUN #17 -- rigid frame, load on top span at mid-point, 12" span.

Weight	Accumulated Weight	Dial Reading	Deflection
0.00	0.00	900	-----
3.45	3.45	890	0.010
8.40	11.85	859	0.041
8.30	20.05	829	0.071
8.30	28.35	778	0.122
8.30	36.65	766	0.134
8.30	44.95	735	0.165
8.20	53.15	704	0.196
8.20	61.35	673	0.227
8.20	69.55	641	0.259
8.20	77.75	599	0.301
8.00*	85.75*	545*	0.355*
8.00	93.75	115	0.785
-----	-----	set 910	-----
8.00	101.75	7.5	0.695
0.00**	101.75**	**	**

(*) denotes formation of the first plastic hinge under the load.
(**) denotes failure of the structure with hinges at the corners.

RUN #18 -- fixed-end beam, span of 12", load at mid-point.

Weight	Accumulated Weight	Dial Readings	Deflection
0.00	0.00	900	-----
3.45	3.45	882	0.018
8.30	11.75	833	0.067
8.30	20.05	785	0.115
8.30	28.35	738	0.162
8.30	36.65	693	0.207
8.30	44.95	650	0.250
8.30	53.05	610	0.290
8.20	61.25	570	0.330
8.20	69.45	530	0.370
8.40	77.85	493	0.407
7.20	85.05	454	0.446
1.20*	86.25*	*	*

(*) denotes failure with the formation of hinges under load
and at the two fixed ends of the beam.

EXPERIMENTAL DATA

RUN #19 -- Fixed-end beam, span of 12", load at mid-point.

Weight	Accumulated Weight	Dial Readings	Deflection
0.00	0.00	800	------
3.45	3.45	792	0.008
8.30	11.75	774	0.026
8.30	20.05	755	0.045
8.30	28.35	736	0.064
8.30	36.65	717	0.083
8.30	44.95	699	0.101
8.10	53.05	681	0.119
8.30	61.35	663	0.137
8.10	69.45	644	0.156
8.40	77.85	625	0.175
8.00	85.85	607	0.193
8.00	93.85	588	0.212
8.00	101.85	570	0.230
8.00*	109.85*	*	*

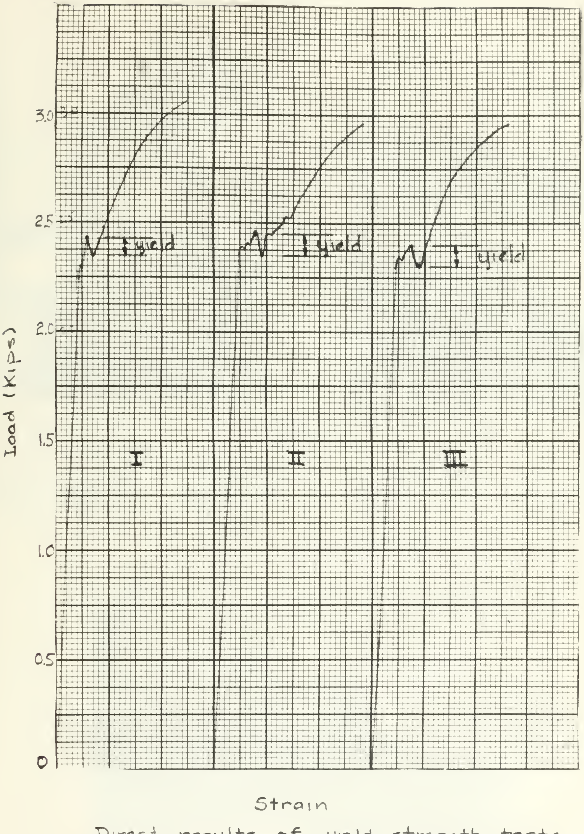
RUN #20 -- Fixed-end beam, span of 12", load at mid-point.

Weight	Accumulated Weight	Dial Readings	Deflection
0.00	0.00	900	------
3.45	3.45	893	0.007
8.40	11.85	874	0.026
8.30	20.15	854	0.046
8.30	28.45	834	0.066
8.30	36.75	814	0.086
8.30	45.05	794	0.106
8.20	53.25	775	0.125
8.20	61.45	756	0.144
8.20	69.65	736	0.164
8.20	77.85	716	0.184
8.00	85.85	694	0.206
8.00	93.85	674	0.226
8.00	101.85	653	0.247
8.00*	109.85*	*	*

(*) Denotes failure with the formation of hinges under load and at the two fixed ends of the beam.

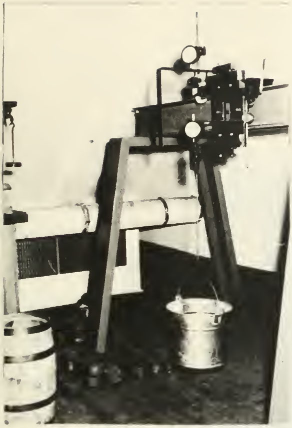
NOTE: RUN #21 was continuous beam over central support with one end fixed, and opposite end of other span simply supported. No deflections were read, but fixed-end span was loaded by the usual increments to failure. The failure load was determined to be 109.35 lbs.

FIGURE 18



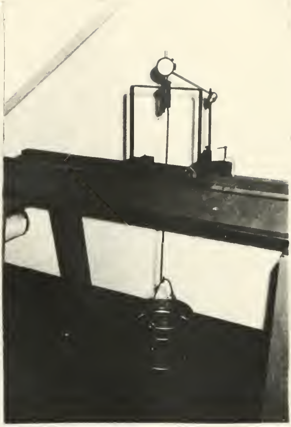
Direct results of yield strength tests on experimental material. With effective speciman area of .0623 in^2 f_s = 38,500 psi.

FIGURE 19



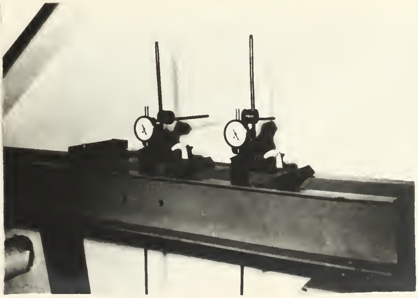
(a)

Photo showing test with side-load on Rigid Frame Note ingots. Also note Frame Tester bolted on Bench.



(b)

Photo showing test with top span load on Rigid Frame. Note Frame Tester on Bench and Load hanger and Bucket.



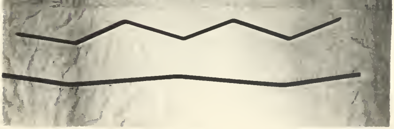
(c)

Photo showing beam test in process. Note position of gages. Note also gage holders and method of clamping them to Bench.

FIGURE 20

(a)

Photo showing 2-span beam test. Note curvature of beam. Note knife-edges.



(b)

Photo showing deformed samples after testing. Note perfectly straight sections between hinges.

Thesis 12850
B94 Butler
An investigation of the formation of plastic hinges in simple structures.